Chapter 1

Basic Monetary Analysis in a General Equilibrium Framework

1.1 The Sidrauski-Brock Model

The objective of this course's first two chapters is to develop and compare two basic dynamic general equilibrium models designed for macroeconomic and monetary analysis. The presentation begins by outlining the simpler of these two frameworks, which involves individual agents with infinite planning horizons, before turning to the more difficult alternative in which agents have finite lifetimes.

The first approach is a discrete-time version of the model developed by Sidrauski (1967) and Brock (1975), which is also presented in Section 4.5 of Blanchard and Fischer's (1989) well-known text. A typical household wants at time t to maximize

$$(1.0) \qquad u(c_t, m_t) + \beta u(c_{t+1}, m_{t+1}) + \beta^2 u(c_{t+2}, m_{t+2}) + \ldots \qquad \beta = \frac{1}{1+\rho}$$

subject to a sequence of constraints like[1]

$$(1.1) \qquad f(k_t) + tr_t = c_t + (1+\nu)k_{t+1} - (1-\delta)k_t + (1+\nu)(1+\pi_t)m_{t+1} - m_t$$

Here c_t, k_t, and m_t are per-capita magnitudes of consumption,[2] capital, and real money balances, with stocks measured at the start of the period indicated. Also, π_t and tr_t are the inflation rate and the amount of lump-sum transfers (net of taxes) to the household per capita while ν is the population growth rate and $\rho > 0$ is a time-preference parameter. The utility and production functions u(c,m) and f(k) are assumed to have positive first

[1] Here the constraint is written in real terms, with income on the left and consumption plus net addition to assets on the right. To illustrate how the terms arise, let N_t be the number of household members in t and note that the household's buildup of money holdings during t is $N_{t+1}P_{t+1}m_{t+1} - N_tP_tm_t$. Then convert to real terms by dividing by P_t and to per-capita terms by dividing by N_t. Since $N_{t+1}/N_t = 1+\nu$ and $P_{t+1}/P_t = 1 + \pi_t$, the final two terms are as shown.

[2] Actually, c_t is an index of the economy's many distinct goods—see Section 1.6 below.

derivatives, negative second derivatives, and to be well behaved.[3] The resulting optimality conditions are (1.1) and

(1.2) $u_1(c_t,m_t) = \lambda_t.$

(1.3) $\beta u_2(c_{t+1},m_{t+1}) = \lambda_t(1+\nu)(1+\pi_t) - \beta\lambda_{t+1}$

(1.4) $(1+\nu)\lambda_t = \beta\lambda_{t+1}\left[f'(k_{t+1})+1 - \delta\right]$

plus two transversality conditions.[4] If the latter are satisfied, the four equations determine time paths for c, m, k, and λ, given π and ν. Then for a competitive market equilibrium (CE) we also have

(1.5) $g_t + tr_t = (1+\nu)(1+\pi_t)m_{t+1} - m_t$

(1.6) $m_t = M_t / P_t$ and

(1.7) $\pi_t = \frac{P_{t+1}-P_t}{P_t}.$

Here g_t is government consumption so (1.5) is the government budget constraint. These seven equations determine c, m, k, λ, π, P, tr for given paths of M and g. Equations (1.1) and (1.5) imply the overall resource constraint $f(k_t) = c_t + (1+\nu)k_{t+1} - (1-\delta)k_t + g_t$; this implication is an example of Walras's Law.

In the foregoing analysis we should actually have included a market for the rental of capital goods. Then the quantity of capital used in production, k^d_t could differ from the amount of capital that the household had at the end of period t−1, and a market-clearing equilibrium condition would be that $\Sigma k_t = \Sigma k^d_t$, where the sums are over all

[3] The production function is well behaved if it satisfies the Inada conditions $f'(0) = \infty$ and $f'(\infty) = 0$. For u(c, m), such conditions apply to the partial derivatives $u_1(c, m)$ and $u_2(c,m)$.
[4] Since (1.2), (1.3), and (1.4) are difference equations, some initial or terminal conditions are needed to determine the time paths uniquely. In this regard the transversality condition or conditions are crucial. This topic will be discussed below.

households. But with all households alike, the latter would imply that $k_t = k^d_t$, so the outcome would be the same as we have obtained. Thus we have ignored the possibility of $k_t \neq k^d_t$ although to do so is actually improper. The same sort of statement applies to our neglect of the labor market. There we are assuming inelastic labor supply by each household, which is special but legitimate. What is not proper is to assume, as we have implicitly done, that labor usage in production by each household is the same as its own labor supply (although this turns out to be the case in equilibrium when all households are alike).

If we added one-period private bonds to the model given above, we would find[5] that $r_t = f'(k_{t+1}) - \delta$ where $1+r_t = (1+R_t)/(1+\pi_t)$ with $1/(1+R_t)$ = the price of a bond in t. Here r_t and R_t are real and nominal rates of interest on these bonds. Since the equilibrium quantity of bonds purchased by a typical household would be zero, the equations given above can be interpreted as pertaining to an economy with private bonds or loans.

For a steady state[6] the foregoing system collapses to

$$f(k) = c + (\nu+\delta)k + g \tag{1.8}$$

$$\frac{u_2(c,m)}{u_1(c,m)} = \frac{(1+\nu)(1+\pi)}{\beta} - 1 \tag{1.9}$$

$$\frac{(1+\nu)}{\beta} = f'(k) + 1 - \delta \tag{1.10}$$

These determine c, m, k for exogenously given values of g and π.

The last of these equations can be written as

[5] See Problem 3 at the end of the chapter.

[6] A steady state is a situation in which every variable grows at some constant rate (possibly zero). With no technical progress, the model at hand implies that any steady state has zero growth for the variables k_t, c_t, and m_t. If exogenous technical progress at a constant rate was assumed, the model would be equivalent to the neoclassical growth model with monetary variables added. The neoclassical growth model is studied in a later chapter (not yet available), along with recent endogenous growth alternatives.

$$(1+\rho)(1+\nu) = f'(k) - \delta + 1$$

or

(1.11) $$\rho + \nu = f'(k) - \delta$$

where we neglect the product $\nu\rho$. So we have the familiar *modified golden rule* condition for the steady-state real return to capital. It says that the CE steady state capital stock is that which makes the net marginal product of capital equal to the rate of time preference plus the rate of growth of total output. (For more discussion, see Chapter 3.)

Now let's turn to monetary relations. Combining (1.9) and (1.10) we get

(1.12) $$\frac{u_2(c,m)}{u_1(c,m)} = \left[f'(k) + 1 - \delta\right](1+\pi) - 1$$

$$= 1 + r + \pi + r\pi - 1$$

$$\doteq r + \pi$$

But the latter could be written as $\Phi(c,m) = R$ or, under fairly general conditions, as

(1.13) $$m = L(c, R)$$

which is like a typical textbook money demand equation. Here it can be shown that $L_1>0$, $L_2>0$ for "reasonable" restrictions on u. If we have u Cobb-Douglas, for example, we get m = const.c/R.

But what about the period-by-period relation? Put (1.4) into (1.3):

(1.14) $$\beta u_2(c_{t+1}, m_{t+1}) = (1+\pi_t)\beta\lambda_{t+1}\left[f'(k_{t+1}) + 1 - \delta\right] - \beta\lambda_{t+1}$$

(1.15) $$\frac{u_2(c_{t+1}, m_{t+1})}{u_1(c_{t+1}, m_{t+1})} = (1+\pi_t)\left[f'(k_{t+1}) + 1 - \delta\right] - 1$$

which is like (1.11) so we get

(1.16) $$m_{t+1} = L(c_{t+1}, R_t).$$

All of this provides some justification for the usual textbook form of a "money demand equation" with (expected) c_{t+1} as the transaction variable. But (1.16) is actually a "portfolio balance" condition; the proper money demand function is of the form $m_{t+1} = \tilde{m}(m_t, k_t, tr_t, tr_{t+1}, ..., \pi_t, \pi_{t+1}, ...)$.

Before continuing let's pause to consider the justification for including m_t in $u(c_t,m_t)$. To do monetary analysis, it is often necessary to distinguish monetary from non-monetary assets. The only sensible way to do so is by means of money's role as the medium of exchange,[7] which implies that it—unlike other assets—is helpful in facilitating transactions. My own preferred parable is that we have

(1.17) $\tilde{u}(c_t,l_t)$ with $l_t = 1 - n_t - \psi(c_t m_t)$

with n_t approximately constant and $\psi_1>0, \psi_2<0$ for $m < m^*$. Here $s_t = \psi(c_t,m_t)$ is time spent "shopping." Then $\tilde{u}(c_t,l_t) = \tilde{u}(c_t, 1-n- \psi(c_t,m_t)) = u(c_t,m_t)$. If we know the properties of $\tilde{u}$ and ψ we know them for $u(c_t,m_t)$. Note that as $m_t \rightarrow 0$, we would *not* expect $u_2 \rightarrow \infty$, since $u_2 = \tilde{u}_2(c_t,l_t)\ \psi(c_t,m_t)$ and $\psi_2(c_t,m_t)=\frac{\partial s_t}{\partial m_t}$ will not $\rightarrow -\infty$ as $m_t \rightarrow 0$. As $m_t \rightarrow 0$, we approach a state of barter but there is still a finite maximum for s_t.

There are several other formulations that are qualitatively rather similar to the shopping time approach. The Baumol-Tobin setup is one, the cash-in-advance constraint is a second, and others have been developed recently. All of them imply that the activity of conducting transactions is facilitated by holding (real) quantities of the medium of exchange (MOE), precisely because it is generally accepted in exchange. This approach

[7] Many assets serve as stores of value while an economy's money is not always its medium of account.

does not tell *which* items will be MOE, i.e., money. We will come back to that issue below.

There are two major questions relating to the steady state properties of any monetary model. One is whether *superneutrality* prevails and the other is what (steady, anticipated) inflation rate is socially optimal. The model considered above has steady state values of m, c, k determined by (1.8) (1.9) (1.10) for given π and g. But these equations are recursive: (1.10) determines k, then (1.8) determines c. So both k and c are independent of π! In that sense the model has the property of superneutrality (even though "shopping time" is affected by π).

But suppose we include n in the utility function, as we should. Then we get

$$f(n,k) = c + (\nu+\delta)k + g \qquad (1.8')$$

$$\frac{u_2(c,m,n)}{u_1(c,m,n)} = \frac{(1+\nu)(1+\pi)}{\beta} - 1 \qquad (1.9')$$

$$\frac{(1+\nu)}{\beta} = f_2(n,k) + 1 - \delta \qquad (1.10')$$

and

$$\frac{u_3(c,m,n)}{u_1(c,m,n)} = -f_1(n,k), \qquad (1.18')$$

the last of which comes from the additional household optimality condition

$$u_3(c_t,m_t,n_t) + \lambda_t f_1(n_t,k_t) = 0. \qquad (1.18)$$

Now, in this case one cannot determine k or c without using all four equations. So k and c do depend upon π. Thus superneutrality does *not* hold in a properly articulated version of the Sidrauski model.

It would hold, however, if $\frac{u_3(c,m,n)}{u_1(c,m,n)}$ were independent of m, as would be true if u were Cobb-Douglas. So superneutrality might be a good approximation. The shopping time setup does not support this notion, however, even if $u = c^{\alpha} l^{1-\alpha}$ and $s = c^{1+\beta} m^{-\beta}$ are of the Cobb-Douglas form.

Let me conclude this section with a bit of elaboration on the shopping-time setup mentioned above. It should be thought of as a parable, but some students introduced to monetary analysis within the last 15 years evidently have trouble accepting it even at that level. So, think of a period as a month. Over a typical month, the various members of a household will make purchases of hundreds of different goods from almost as many different sellers [see Lucas (1980)]. On each purchase, if the family member makes the payment with money then the transaction goes smoothly and quickly. If he doesn't have enough money with him, or chooses not to use it, then he has to arrange a credit purchase. Sometimes this goes smoothly and takes only a little longer than making the transaction with money. But sometimes the seller refuses to sell on credit and often it takes a considerable amount of time to convince the seller that the family member is a good credit risk. On each of these occasions extra time must be spent to make the purchase—perhaps even an extra shopping trip to the seller's location. All of these purchases are distributed randomly across time within the month and perhaps randomly across members of the family. [Income is arriving at random times during the month, most in money form, some not. Conversions have fixed costs or require a trip.]

Now suppose that the household makes purchases of about $10,000 per month. If on average the household's money balances are only $10, then there will be many purchases requiring credit arrangements (and extra time). If the average money holdings are \$100 the number of difficult purchases will be fewer, and if average money holdings are \$1000 the number will be small. At some finite figure the household will be satiated with the services of money holdings. But it is unclear what that figure would be. It might even be larger than \$10,000 because that is split among different members of the household. But the larger the average amount of money held, the greater the savings of shopping time will be—up to the satiation level.

The foregoing describes the relevant level of money holdings as the average over the period. That does not agree with out model, in which the relevant magnitude is the stock at the first of the month. But a period analysis recognizes only two possibilities—at the beginning and end of the period. And all of the analysis that we have done could just as well use the end-of-period magnitude as the relevant one. Indeed, in the McCallum and Goodfriend *New Palgrave* article on the demand for money (1987)—and in much recent formal analysis—the end-of-period value of m_t is used. But the start-of-period convention is more agreeable to some analysts (for cash-in-advance reasons) so that's the way the model is written here.

In recent years there has been much work on the development of explicit microeconomic setups that would rationalize the macroeconomic role of a medium of exchange. Notable papers have been produced by Kiyotaki and Wright (1989), Ireland (1994), Lacker and Schreft (1996), and others. Such work is certainly useful, but it seems unnecessary to go through such a procedure in each paper that features macroeconomic analysis for a monetary economy. If the MOE were gold, it would not be necessary to explain why that particular metal provided the monetary standard in analyses concerned with the workings of the gold standard.

1.2 Optimal Inflation Rate

Now let's turn to welfare analysis. The social (Pareto) welfare problem at time $t = 1$ is to maximize $u(c_1,m_1,n_1) + \beta u(c_2,m_2,n_2) + \cdots$ subject to constraints like

(1.19) $$f(n_t,k_t) = c_t + (1+\nu)k_{t+1} - (1-\delta)k_t + g_t .$$

For simplicity let's set $g_t = 0$, since g plays no role in the setup. The first-order conditions (FOCs) are (1.19) and

(1.20) $$(1+\nu)\frac{u_1(c_t,m_t,n_t)}{u_1(c_{t+1},m_{t+1},n_{t+1})} = \beta[f_2(n_{t+1},k_{t+1}) + 1 - \delta]$$

$$\Rightarrow (1+\nu) + \beta[f_2(n,k) + 1 - \delta]$$

in the steady state, plus

(1.21) $$\frac{u_3(c_t,m_t,n_t)}{u_1(c_t,m_t,n_t)} = -f_1(n_t,k_t) \quad \Rightarrow \frac{u_3(c,m,n)}{u_1(c,m,n)} = -f_1(n,k)$$

and

(1.22) $$u_2(c_t,m_t,n_t) = 0$$

$$\Rightarrow u_2(c,m,n) = 0.$$

[Here we are assuming that $u(c_t,m_t,n_t)$ permits satiation with respect to m_t, that is, that $u_2 \leq 0$ for m_t greater than some satiation level (which could be related to c_t), with $mu_2(c,m,n) = 0$.] But looking back, the C.E. relations (1.8′) – (1.10′) plus (1.18′) and (1.10′) imply that (1.22) will hold *only if*

(1.23) $$\left[f_2(n_{t+1},k_{t+1}) + 1-\delta\right](1+\pi_t) - 1 = 0.$$

Thus Pareto optimality of a steady state requires that $1 + (f_2 - \delta) + \pi + (f_2 - \delta)\pi - 1 = 0$ or, to a good approximation, that $\pi_t = -[f_2(n_{t+1}, k_{t+1}) - \delta]$.

Alternatively, the condition is $R_t = 0$. The common sense of this famous result is clear. As long as $\pi > -(f_2 - \delta)$ or $R > 0$, reducing π will increase equilibrium m and reduce shopping time, thereby increasing utility. But reducing π requires no resources, so m should be increased to the point at which agents are satiated with the transaction facilitating *services* of money. That is the essence of Friedman's (1969) argument.

It is important to note that this result does not depend upon superneutrality (which does not prevail in our model). That needs to be mentioned because statements to the contrary have been made by prominent economists. For example, Blanchard and Fischer (1989) conduct an optimal inflation analysis on their pp. 191-193 in a Sidrauski-type model with superneutrality and obtain the Chicago Rule. But then they state (on p. 191) that "The super-neutrality result has direct implications for the optimal rate of money growth [i.e., inflation]. *Because money growth does not affect real consumption in the steady state* (emphasis added), the steady state utility is maximized by making real balances large enough that their marginal utility equals zero…this implies having a rate of deflation equal to the real rate of interest… Thus in this model we obtain the Friedman result that it is optimal to satiate individuals with money and that the rate of return on money should be the same as on capital."

Then they go on to say "These results about the effects of money growth on capital accumulation and welfare differ from those derived in previous sections." And on p. 192 they say "In the model of Section 4.3 money growth affects capital accumulation; in the [Sidrauski] model it does not. So where does the difference come from?" Their answer is "finite lifetimes." So they seem to be suggesting that the assumption of finite lifetimes prevents superneutrality and also implies that the Chicago Rule $\pi = -(f_2 - \delta)$ will be incorrect.

Indeed, on p. 181 they say that the Chicago Rule is not optimal in a Baumol-Tobin OG model when the steady state value of $r = f_2 - \delta$ is positive. Specifically, they say that when $r > 0$, "it is optimal to have positive capital accumulation, but if the nominal rate of interest is equal to zero (plus epsilon), individuals hold all their wealth in the form of money and the aggregate capital stock will be equal to zero." This seems confused. Why should not some fraction of wealth be in the form of capital when $r > 0$ and $r + \pi = 0$? Indeed, as $k \to 0$, $r \to \infty$ in most models!

In any event, it is clearly wrong to believe that the absence of superneutrality means that the Chicago Rule is inapplicable. We have already shown that, in our Pareto optimality analysis of the Sidrauski model with $u(c_t,m_t,n_t)$. Also, Friedman's "satiation" logic seems intuitively correct.

So where does the Blanchard and Fischer reasoning go astray? The problem lies in their *approach* to optimality. They proceed by finding $u(\pi)$ for steady states and then maximizing with respect to π. But we know from elementary growth theory—and from Blanchard and Fischer's Chapter 2 (p. 45)—that this is incorrect. To see this, delete m from our model. Then in any steady state we have $c = f(n,k) - (\nu+\delta)k - g$ so

(1.24) $$u(c,n) = u[(f(n,k) - (\nu+\delta)k - g), n] \quad \text{and}$$

(1.25) $$\partial u/\partial k = u_1(c,n)[f_2(n,k) - (\nu+\delta)].$$

So setting the latter equal to zero yields

(1.26) $$f_2(n,k) - \delta = \nu,$$

which is known as the "golden rule" condition. But the optimal steady state k is given by the *modified* golden rule condition

$$(1.27) \qquad f_2(n,k) - \delta = \nu + \rho, \quad \text{where} \quad \beta = \frac{1}{1+\rho}.$$

That proposition most readers will know—but in any case we have already proved it for the Sidrauski model because we know that P.O. implies (1.20) and the steady state version of that equation is $(1 + \nu) = \beta[f_2(n,k) + 1 - \delta]$ or

$$(1.28) \qquad (1+\rho)(1+\nu) = f_2 - \delta + 1, \quad \text{or}$$

$$(1.29) \qquad \rho + \nu \doteq f_2 - \delta.$$

Thus Blanchard and Fischer have in effect solved the wrong problem.[8]

1.3 Comments on Approach

I personally believe that the foregoing presentation provides a fair picture of "the current mainstream approach to money-demand theory." There is nothing in Blanchard and Fischer's chapter 4 that would contradict that opinion, for they strongly criticize the other frameworks that they use—especially the OG model in Section 4.1. (A review of their Chapter 4 appears below.) This claim was put forth in my paper with Goodfriend in our *New Palgrave* article on money demand theory. Perhaps the claim looks suspicious to some readers because we use it in an exposition that features the shopping-time setup. But, as we briefly explain, what we mean is that this setup is one that represents *transaction oriented* theories. Others that do so are Money-In-Utility Function, Cash-In-Advance, Baumol-Tobin, and models with a transaction-cost term in the budget constraint. It is my belief that the shopping-time setup is somewhat better than the others

[8] On this topic, see Chapter 3.

for various reasons. It is a bit more explicit than the MIUF setup and enables one to think about limiting properties more clearly. The CIA constraint would apply to periods that are too short to be interesting. The transaction-cost in the budget constraint setup is one that is more complicated from a general equilibrium viewpoint—resource payments are being made to some firm that sells transaction-facilitating services, so these should be made part of the model but usually are not. The same applies to the Baumol-Tobin model. But the main point is that these are all parables or metaphors for an approach that features money as an asset that (unlike other assets) facilitates transactions. On this topic, see Feenstra (1986).

Now, to conduct monetary theory in models with more than one paper asset there must be *some* analytical feature that distinguishes money from the others. But by the definition of money that distinction must be that money does, and the others do not, function as a medium of exchange. (All assets are stores of value and the medium of account need not be the same as the MOE.) So there must be a feature in the model to reflect this distinction. Of course, some models have only one paper asset but then you can tell whether it is money or bonds by the presence or absence of this feature.

There are several other general approaches to modeling money's role besides the transaction approach. One is Tobin's portfolio balance approach that uses rate-of-return uncertainty as the basis for distinguishing between monetary and non-monetary assets. This one attracts little attention today. There are two major problems: agents care about real returns, not nominal returns; and there are many assets just as safe as money (i.e., with the same risk properties) that pay higher interest, so this feature cannot explain why money is held.

The other main approach has been the store-of-value approach provided by overlapping-generation (OG) models in which money is held because it makes possible exchanges across generations. But in reality there are many other stores of value so this is unsatisfactory for monetary theory. There is, however, no reason not to use an OG model with a shopping-time technology, so we shall do that in the next chapter.

1.4 Ricardian Equivalence

An important and controversial topic is the Ricardian Equivalence Proposition, which concerns the importance or unimportance of fiscal deficits. The precise nature of this proposition is developed in Problems 5-7, at the end of this chapter.

1.5 Transversality Conditions

It is mentioned above that there are additional conditions, called transversality conditions, relevant to household optimality. These are discussed in Problem 12 and in Appendix A of Chapter 3.

1.6 CES Consumption Index and Monopolistic Competition

Before concluding, we want to recognize that the consumption quantities considered in Sections 1.1-1.3 can legitimately be interpreted as indexes reflecting bundles containing many distinct goods. For an individual household that consumes n distinct goods, consider the consumption index

$$(1.30)\quad c = n^{1/(1-\theta)} \left[\sum c_i^{(\theta-1)/\theta}\right]^{\theta/(\theta-1)} \qquad \theta>1$$

where the summation is over goods i = 1,2,…,n and θ is the elasticity of substitution in consumption. Our object here is to show that, with this index, the solution to the simplified problem

$$(1.31)\quad \text{maximize} \quad u(c) \quad \text{subject to} \quad pc = y$$

is fully consistent with the solution to the more complete problem

(1.32) maximize $u(n^{1/(1-\theta)}\,[\sum c_i^{(\theta-1)/\theta}]^{\theta/(\theta-1)})$ subject to $\sum p_i c_i = y$.

Such consistency requires that there exist a suitable definition of p that satisfies $pc = \sum p_i c_i$ and also that the Lagrange multipliers on the budget constraints of the two solutions are the same.

For problem (1.31) write the Lagrangian expression $u(c) + \lambda_2[y - pc]$; it yields the first-order condition

(1.33) $u'(c) = \lambda_2 p$.

Next, write an analogous Lagrangian expression for problem (1.32). The typical first-order condition, relevant for c_i, is

(1.34) $u'(\ )\, n^{1/(1-\theta)}\,[\sum c_i^{(\theta-1)/\theta}]^{1/(\theta-1)}\, c_i^{-1/\theta} - \lambda_3\, p_i = 0$

Rearranging and using (1.30), we obtain

(1.35) $c_i = \dfrac{cn^{-1}}{p_i^{\theta}}\,(u'(.)/\lambda_3)^{\theta}$.

Thus, using (1.33) and imposing $\lambda_2 = \lambda_3$, we have

(1.36) $c_i = c\, n^{-1}\,(p/p_i)^{\theta}$.

Now substitute (1.36) into $pc = \sum p_i c_i$ to obtain

(1.37) $pc = \sum p_i\, c\, n^{-1}\, p^{\theta}/p_i^{\theta} = c\, n^{-1}\, p^{\theta} \sum p_i^{1-\theta}$.

Finally, cancelling c and solving for p yields

(1.38) $p = (n^{-1} \sum p_i^{1-\theta})^{1/1-\theta}$.

This is the definition of p that is needed, so the demonstration is complete.

It is satisfying to note that, if $p_i = \overline{p}$ for all i, we have

(1.39) $p = (n^{-1} \sum \overline{p}^{\,1-\theta})^{1/(1-\theta)} = (n^{-1}\, n\, \overline{p}^{\,1-\theta})^{1/(1-\theta)} = \overline{p}$, and in that case $c_i = cn^{-1} (\bar{p}/\bar{p})^{\theta} = c/n$. Also, it is notable that (1.36) shows that the typical household has a demand function for each distinct good that has elasticity $-\theta$ with respect to that good's relative price. Thus, with households alike, total demands for each good will also have that elasticity. (The development of this index is usually attributed to Dixit and Stiglitz (AER, 1977).)

The foregoing development can be used to modify the model so as to suppose that each household specializes in the production of some particular good and possesses some market power in selling that good. This topic is discussed in Problem 18.

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Problems

1. By definition, a steady state is a path along which every variable grows at some constant rate (possibly zero or negative). Show that in the model of eqns (1.1)-(1.7), the variables k_t, c_t, and m_t must be constant in a steady state. [It may be helpful to use the fact that if $Y_t = X_t + Z_t$ for any (nonzero) variables X, Y, and Z, then all three variables can grow at constant rates only if the rate is the same for all three.]

2. Suppose that an economy's production function is altered by an invention that increases the importance of capital relative to labor in the production of goods. What will be the effect on the steady-state rate of return on capital, according to the Sidrauski-Brock model?

3. Note the statement about private one-period bonds made in the middle of page 3. Then introduce instead one-period government bonds. Let B_{t+1} be the number of government bonds purchased by a typical household during period t, each of which costs $Q_t = 1/(1+R_t)$ dollars and is redeemed in t+1 for \$1. Define $b_t = B_t/P_t$ and then generalize the household and government budget constraints (1.1) and (1.5). Finally, verify the claim about private bonds.

4. Suppose that the typical household maximizes $u(c_t, m_t) + \beta(1+\nu)^{\phi} u(c_{t+1}, m_{t+1}) + \beta^2(1+\nu)^{2\phi} u(c_{t+2},m_{t+2}) + \ldots$ instead of the expression above (1.1). What is the interpretation in the case $\phi = 1$? (Assume $\beta(1+\nu)<1$.) How would that change in setup affect the steady-state rate of return on capital? Would it increase or decrease steady-state output per person?

5. The famous "Ricardian equivalence proposition" asserts that bond-financed changes in the government's net tax collections ($-tr_t$), with unchanged paths of g_t and M_t, will have no effect on most variables of macroeconomic interest, including c_t, R_t, r_t, and P_t. Show that this property holds in the model of Section 1.1, extended so as to include government bonds as an additional asset. [Hint: show that values for c_t, m_t, k_{t+1}, and $r_t = f'(k_{t+1}) - \delta$ can be determined without reference to b_t or tr_t, when Walras' Law is used to eliminate the household budget constraint.] [For simplicity, let $\delta = \nu = 0$.]

6. Does the Ricardian result hold if per-capita labor time is included in the utility function, as on pages 6-7?

7. Does it hold if lump-sum taxes $-tr_t$ are replaced with taxes on production, $\tau_t f(n_t,k_t)$?

8. (Monopolistic Competition) Suppose that a typical household is the only producer of a particular good, one of the goods that are included in the Dixit-Stiglitz (D-S) bundle desired and consumed by each household. Each household's demand

for good i is proportional to $(P^i_t/P^A_t)^{-\theta}$, where P^i_t/P^A_t is the relative price (with P^A_t the D-S price index for goods in general) and θ the elasticity of substitution in consumption among the various goods. Thus each household faces a demand function for its own good of the form $d_t = d^A_t\,(P_t/P^A_t)^{-\theta}$ where d^A_t is the D-S aggregate quantity consumed of all goods. We assume that the typical household being considered produces and sells the quantity demanded of its good, so that

$$(1)\ \ f(n_t, k_t) = d^A_t\,(P_t/P^A_t)^{-\theta}.$$

Its budget constraint is

$$(2)\ \ d^A_t\,(P_t/P^A_t)^{-\theta}(P_t/P^A_t) + tr_t = c_t + k_{t+1} - (1-\delta)k_t + (1+\pi_t)m_{t+1} - m_t$$

and its utility function is

$$(3)\ \ u(c_t, m_t, 1 - n_t) + \beta\, u(c_{t+1}, m_{t+1}, 1 - n_{t+1}) + \beta^2\, u(c_{t+2}, m_{t+2}, 1 - n_{t+2}) + \ldots$$

(i) In this setting, the household's choice variables include c_t, k_{t+1}, m_{t+1}, n_t, and P_t plus the Lagrange multipliers λ_t and ξ_t (the latter pertaining to (1)). What are the conditions necessary for household optimality?
(ii) For general equilibrium, there are the usual market-clearing conditions that must be satisfied. In addition, under the "symmetry" assumption that the same form of demand and production function pertains to each household, we will have $P_t/P^A_t = 1$ as an equilibrium condition. What then will be the relationship between the marginal product of labor and the marginal utility of leisure, when expressed in comparable units? How does this relationship depend on θ?

9. Show that the "Chicago Rule" for the socially-optimal steady inflation rate is applicable in a modified version of the Sidrauski model that incorporates eq. (1.17) with n_t variable.

10. What is evidently being assumed about ψ_2 and ψ_{22}?

11. Reconsider the optimal inflation analysis by using a "cash in advance" specification that requires each household's consumption to satisfy
$c_t \leq m_t$ in each period. In doing this assume that labor supply is variable so that the production and utility functions are $f(n_t, k_t)$ and $u(c_t, 1\text{-}n_t) + \beta\, u(c_{t+1}, 1\text{-}n_{t+1}) + \ldots$.

For convenience, let $g_t = 0$.

12. Consider a household that seeks at time t = 1 to maximize

$$u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \ldots + \beta^{T-1} u(c_T)$$

subject to

$$f(k_t) = c_t + (1+\nu)k_{t+1} - (1-\delta)k_t \qquad \text{for } t = 1, 2, \ldots, T.$$

Find the household's first-order conditions. What do these suggest for the case in which $T \to \infty$?

13. Now consider a special case with $u(c_t) = \log c_t$, $f(k_t) = Ak_t^{\alpha}$, and $\delta=1$. For the case with $T \to \infty$, find a solution path for k_t and c_t. Hint: guess that current choices are log-linear functions of the current "state" of the system.

14. Let $\alpha = 1.0$ in problem 13 and find a solution to the first-order conditions that does not satisfy the transversality condition suggested in problem 12. (Hint: Treat the FOC's as a system of linear difference equations.) (To simplify notation, let $\nu = 0$.)

15. Consider an economy in which a typical household maximizes $u(c_1) + \beta u(c^2) + \ldots$subject to

$$(1-\tau_t)f(k_t) + tr_t = c_t + (1+\nu)k_{t+1} - (1-\delta)k_t.$$

The government does not consume any goods itself but it taxes production at the rate τ_t and returns the proceeds to households by means of lump sum transfers (tr_t per person). Find conditions for a competitive equilibrium (CE) and determine whether it is socially optimal.

16. Now specialize by setting $u(c_t) = \log c_t$, $f(k_t) = Ak_t^{\alpha}$, and $\delta = 1.0$. Determine whether the CE solution path is dynamically stable.

17. Consider steady states (in the economy of problem 15) that are feasible but are not necessarily competitive equilibria. Find the marginal product of capital that yields the highest steady state level of consumption per person. (This is known as the "golden rule" steady state.) Next find the value of τ that would induce a CE to approach that rate of consumption. Is that value of τ socially optimal? Explain.