Real Options and Merchant Operations of Energy and Other Commodities

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Abstract

The value chain for energy and other commodities entails physical conversions through refineries, power plants, storage facilities, and transportation and other capital-intensive infrastructure. When the operation of such commodity conversion assets occurs alongside liquid markets for the input and output commodities, the operating flexibility of conversion assets can be managed as real options on the underlying commodity prices. Merchant operations is an integrated trading and operations approach that (i) buys and sells commodities to support market-value maximizing operating policies and (ii) values conversion assets, for capital budgeting and trading purposes, based on the cash flows such policies produce. This monograph provides a unique integrated finance and operations perspective on the topic of merchant operations. In particular, this monograph introduces the concept of merchant operations; presents the basic principles of option valuation;
surveys foundational models of commodity and energy price evolution; analyzes the structure of optimal operating policies for commodity conversions, focusing specifically on inventory and other intertemporal linkages in storage, inventory acquisition and disposal, and swing assets; considers a variety of heuristic storage operating policies; and discusses future trends in this multidisciplinary area of research and business applications.
In honor of the late Professor Paul R. Kleindorfer
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Aim and Scope

Commodity conversion assets perform various transformation processes, including the production, refining, industrial and commercial consumption, and distribution of physical commodities and energy sources, such as grains, metals, electricity, coal, crude oil, and natural gas. This monograph deals with the management and valuation of conversion assets. Commodities and energy are traded on physical markets. It is thus natural to approach the management of commodity conversion assets from a merchant perspective, which adjusts the level of the conversion activities to profit from the dynamics of commodity prices. We introduce the expression \textit{merchant operations} to describe this approach.

Implementing merchant operations to maximize the market value of commodity conversion assets is a complex task. It requires both models of the evolution of commodity prices and stochastic optimization models of the conversion activities. The existence of traded contracts on commodities and energy sources allows commodity conversion assets to be interpreted as real options on the prices of the underlying commodities. This real option interpretation greatly facilitates formulating the necessary mathematical models.
The valuation of real options shares the same theoretical foundations as the valuation of financial options. However, the real options that arise in the context of merchant operations are distinguished from financial options by one or more of the following features: (i) decisions at multiple dates, (ii) intertemporal linkages across decisions, (iii) multiple underlying variables, (iv) payoffs determined by operational costs and contractual provisions, (v) engineering-based constraints on operating decisions, and (vi) quantity decisions rather than binary exercise/no-exercise decisions. American and Bermudan financial options also involve decisions at multiple dates with intertemporal linkages, but the other features in this list are largely unique to merchant operations. Intertemporal linkages are particularly important in merchant operations and are related to inventory, the scale of operations, and delays and costs incurred when switching between operating modes. Moreover, even when the structure of an optimal operating (exercise) policy can be explicitly characterized, determining such a policy typically involves numerical computation and approximations. Closed-form solutions are rare in merchant operations, while they are more widespread for financial options.

The aim of this monograph is to present the basic tenets of merchant operations, that is, the management of commodity conversion assets as real options on commodity prices. The scope of this monograph is on the foundational principles that underlie the valuation of real options, on basic models of both the evolution of commodity and energy prices and commodity conversion assets, and on optimal and heuristic operating policies with a focus on commodity storage assets.

A unique aspect of merchant operations is the integration of financial and operational management aspects. This monograph reflects this integrated perspective. Chapter 2 introduces commodity conversion assets and merchant operations. Chapters 3 and 4 discuss the valuation of commodity options and models of the evolution of commodity and energy prices, respectively. Chapters 5 and 6 describe the merchant management of commodity storage assets. Chapter 7 deals with inventory disposal/acquisition and swing assets. Future trends in merchant operations research and applications are discussed in Chapter 8.
This chapter briefly introduces the basic commodity groups in §2.1, and illustrates the trading of commodities in physical and financial markets in §2.2. This discussion provides the necessary elements for introducing the concepts of commodity conversion assets and merchant operations in §2.3. This chapter also shows that merchant operations can be usefully cast in the framework of real options in §2.4, providing some examples of the real option management of commodity conversion assets. Some of these examples are substantial simplifications of how commodity conversion assets are operated in practice, but they set the stage for more realistic models. The discussion of these examples provides a point of departure for the rest of this monograph, which is outlined in §2.5. Section 2.6 offers pointers to the existing literature.

2.1 Commodity Conversion Assets and Merchant Operations

According to the Webster’s New Universal Unabridged Dictionary [223], a commodity is “any unprocessed or partially processed good.” Commodities can be grouped according to three basic categories: agricultural, metals, and energy sources. Each of the groups includes sev-
eral commodity types, for example:

- **Agriculturals**: grains (corn, oats, rice, and wheat), oil and meal (soybean, soyoil, and soymeal), livestock (pork and beef), foodstuff (cocoa, coffee, orange juice, potatoes, and sugar), textiles (cotton), and forest products (lumber and pulp).
- **Metals**: gold, silver, platinum, palladium, copper, and aluminum.
- **Energy sources**: coal, crude oil, heating oil, gasoline, natural gas, propane gas, and electricity.

Commodity types can be further categorized according to grade and quality. In addition, due to limited transportation and storage capacity, the same commodity type at different locations and/or times effectively constitutes separate commodities. For instance, consider natural gas at two ends of a pipeline that transports it, or the availability of natural gas in a storage facility at two different dates. Thus, location and time are defining attributes of commodity types.

Commodities are basic inputs to production, distribution, and consumption activities. They thus play important economic roles. For instance, natural gas is extensively used for heating (e.g., by about 50% of the households in the United States; Casselman [50]) and electricity generation (e.g., about 30% of worldwide electricity production is based on natural gas; Geman [101]). It is increasingly the fuel of choice for new electricity generation projects, due in part to the recent natural gas shale boom (Smith [202]). The increased use of natural gas is having an impact on the worldwide liquefied natural gas (LNG) trade (Davis and Gold [66]) and the planning of new major pipeline construction (Chazan [56], Gold [108]). Natural gas is also an input to several manufacturing processes, including the refining of oil, and the production of chemicals, ammonia and methanol, steel and aluminum, and paper (Kaminski and Prevatt [132]).

Although energy is just one commodity group, it is emphasized in this monograph because of its economic importance. There is also a substantial literature that deals specifically with the application of real option methods to energy.
2.2 Physical and Financial Markets

Commodities are traded in physical and financial markets. In both cases, transactions can be on a spot or forward basis. Spot transactions involve the immediate, or at least very near term (e.g., next day), transfer of the physical commodity or financial ownership thereof. With forward transactions this transfer occurs at a specified date in the future.

Physical trading eventually leads to the transfer of a commodity from one party to another party. For example, the same lot of natural gas for next day delivery may be purchased and sold several times during a given day, but this lot must be transferred from a seller to a buyer on the delivery date.

Financial markets for commodities, such as the Chicago Mercantile Exchange (CME), the New York Mercantile Exchange (NYMEX), the IntercontinentalExchange (ICE), and the London Metal Exchange (LME) trade various contracts that specify different commodity ownership structures. Futures contracts specify obligations to deliver or receive a given amount of a commodity, at a given price, and at a given maturity. (One month maturity futures typically have the highest trading volume.) Most futures traders typically close out their positions via an offsetting trade before maturity. Thus, futures trading need not lead to physical handling of a commodity. Futures prices are set for each delivery date to make the value of the futures contract zero. Hence, there is a single futures price for each delivery date.

Options contracts on futures specify the right to purchase or sell a given futures contracts at a given price and maturity. Call and put options on futures, respectively, give their owners the right to purchase and sell a futures at a contractually specified strike price. The payoff of a call option is the positive part of the difference between the futures price at the option maturity and the strike price. The payoff of a put option is the positive part of the difference between the strike price and the futures price at maturity. Typically, there are options traded with many strikes and expiries.

Additional types of options include calendar spread options on futures contracts. These options give their owners the right to exchange
a futures contract with a given maturity for a futures contract with a different maturity at a given strike price. Effectively, these options let investors buy or sell the difference between two futures prices at a given maturity and strike price. The payoff of a call calendar spread option is the maximum between zero and the difference between the spread between the two futures prices and the strike price. The payoff of a put spread option is defined in an analogous manner.

The options described so far are of the European types, meaning that contractually they can be exercised only at maturity. American (respectively, Bermudan) options can be exercised at any time (respectively, a set of predetermined times) before maturity.

Futures, call and put options, and call and put calendar spread options are traded on organized financial exchanges, such as CME, ICE, LME, and NYMEX, which guarantee the clearing of every trade. That is, these trades are insured by the exchange against the risk of counterparty default. These contracts and additional contracts are also traded on over-the-counter (OTC) markets. Trades on OTC markets are directly exposed to greater counterparty default risk.

Physical trading has an important role in the functioning of production, distribution, and consumption processes. The prices of commodities and contracts on commodities are variable and uncertain. They exhibit both seasonality (deterministic variability) and volatility (stochastic variability). Some theories explain the existence of financial (futures) markets based on price-risk aversion/control arguments (see, e.g., Keynes [138], Duffie [82]). Others rely on transaction costs principles (Williams [224, 225]). In the context of this monograph, financial markets play a useful role in terms of the management of commodity conversion assets in the face of variable and uncertain commodity prices, as explained in §§2.3-2.4.

2.3 Conversion Assets and Merchant Operations

Commodities are produced and used for further processing, consumption, and distribution. In this monograph, the industrial facilities that perform these transformation processes are called commodity conversion assets. Conversion here is used in a broad sense. It refers to
• The production of a commodity, such as the extraction of natural gas from underground wells;
• A physical transformation of a commodity, such as the refining of crude oil, into another commodity, such as gasoline, jet fuel, and naphtha; or
• A change in availability of a commodity, such as the transportation of this commodity between different locations or its storage over time.

Managing commodity conversion assets thus requires performing one or more operational activities, such as the production, processing, refining, transportation, storage, distribution, and physical trading of commodities.

Consider, for example, a storage asset, such as a grain elevator or a metal warehouse, or an underground natural gas storage facility. Storage is used to help match the supply and demand of a given commodity over time. Storage assets feature limited space, and may also have limits on the amount of inventory that can be acquired or disposed of during a given time period. Managing a commodity storage asset requires an inventory trading policy given any space and possible inventory-adjustment (flow) constraints.

As another example, consider a refining asset, such as a crude oil refinery. A crude oil refinery includes various storage tanks for both the inputs, that is, various grades of crude oil, and the intermediate products and outputs. Managing a refinery requires a joint inventory and production policy that determines the inventory levels in these tanks and how the inputs are converted and blended into refined products. In addition, shipping and transportation policies support the sourcing of crude oil and the distribution of refined products.

In practice, the owners of the capacity of a given commodity conversion asset may rent all or part of this capacity to third parties, such as commodity and energy merchants, for given time periods. In particular, this temporary transfer of control of the capacity of a commodity conversion asset occurs when commodity conversion assets are subject to government regulation. This is the case in the natural gas industry in the United States, where, by federal regulation, the owners
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of interstate pipelines and storage facilities must make their capacity available to shippers on an open access basis. Tolling agreements play a similar role in the power generation industry whereby physical generator owners can sell off generation revenue to investors. In this monograph, contracts on the capacity of commodity conversion assets are themselves considered commodity conversion assets. In addition to determining an optimal operating policy for such assets, the determination of their market value is an important practical problem, as it forms the basis for contract negotiation.

Commodity conversion assets are embedded in a market environment for the commodities that they convert. Indeed, commodity conversion assets are the building blocks of both physical and financial commodity markets. Commodity production and storage assets, as well as transportation assets, such as pipelines, ships, and related loading and unloading facilities at ports, play a central role in the determination of spot and futures prices (Williams and Wright [226]). Commodity conversion assets can be managed taking a merchant perspective, which involves the adjustment of the level of conversion activities to reflect the dynamics of commodity prices. For example, the decision to refine crude oil into refined products depends on the respective spreads between the prices of refined products and the price of crude oil. We propose the expression merchants operations to refer to this approach to the management of commodity conversion assets.

The optimal management of merchant operations requires managing operational activities in the face of variable and uncertain commodity prices. The valuation of these operational policies also is important. In general, these are difficult tasks. Interpreting conversion assets as real options facilitates the execution of these tasks.

2.4 Commodity Conversion Assets as Real Options

Managerial flexibility in conversion assets can be viewed as embedded real options on the uncertain evolution of future commodity and energy prices. For example, the ability of oil and natural gas producers to drill or shut down wells depending on the prevailing oil and natural gas prices can be interpreted as a real option on these prices. Other
commodity conversion assets have analogous interpretations as real options on commodity prices. This means that the merchant management of commodity conversion assets amounts to optimally exercising specific real options. To make this concept more concrete, consider some examples.

**Simplified commodity production assets and call options.** Consider a natural gas well. Let \( c \) be the cost of producing one unit of natural gas; \( s_T \) be the spot price of natural gas at time \( T \); and \( Q \) the rate of production. For simplicity, ignore the cost of shutting down and resuming production, the time required to do so, and the ability to time the sale of natural gas. The optimal merchant management policy for this simplified natural gas production asset at time \( T \) is to produce and sell \( Q \) units of natural gas when the spot price of natural gas \( s_T \) exceeds the marginal production cost \( c \), and do nothing otherwise. The payoff of this policy at time \( T \) is

\[
\max\{s_T - c, 0\} \cdot Q,
\]

(2.1)

which is also the payoff of \( Q \) call options on the time \( T \) spot price of natural gas \( s_T \) with strike price equal to \( c \). This observation justifies thinking of the simplified natural gas production asset as a sequence over time of real call options on the natural gas spot price with strike price equal to the marginal production cost and notional amount equal to the production rate.\(^1\) The merchant operations of this asset thus amounts to the optimal exercise of this sequence of options.

The time \( T \) futures price, \( F_{T,T} \), for a futures contract with maturity \( T \) is identical to the time \( T \) spot price, \( s_T \). The payoff (2.1) is thus also the payoff of \( Q \) call options on the time \( T \) futures with maturity \( T \) and strike price \( c \). As discussed in §2.2, such options are traded on exchanges, such as NYMEX. Hence, the current market value of this simplified commodity production asset, when managed at time \( T \) as a merchant operations, can be observed from NYMEX.

The basic principle underlying option valuation can be illustrated by considering a European call option on a futures contract. This dis-

\(^1\)The notional amount of an option is the “number” of such options owned by a trader.
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cussion introduces the ideas of replication and risk-neutral valuation, which are explained in detail in Chapter 3 and play an important role in merchant operations.

Consider the call option on the time $T$ futures price with expiry on date $T$, $F_{T,T}$, and strike price equal to $c$. Assume a binomial process for the futures price dynamics: at time $T$, the initial time 0 futures price $F_{0,T}$ can move up to $u \cdot F_{0,T}$ or down to $d \cdot F_{0,T}$, with $0 < d < 1 < u$. As will become clear shortly, the actual probabilities of the up and down moves are irrelevant. The payoffs of the call option in the up and down states are

$$\text{call}_{T,u,c} = \max\{u \cdot F_{0,T} - c, 0\},$$
$$\text{call}_{T,d,c} = \max\{d \cdot F_{0,T} - c, 0\}.$$

These payoffs can be replicated by setting up on date 0 a portfolio of $Y_F$ dollars worth of futures (that is, the futures contract position notional $n_F$ is $Y_F/F_{0,T}$) and $Y_B$ dollars worth of a risk-free bond. The futures payoffs in the up and down states are

$$(u \cdot F_{0,T} - F_{0,T})n_F = (u - 1)F_{0,T} \frac{Y_F}{F_{0,T}} = (u - 1)Y_F,$$
$$(d \cdot F_{0,T} - F_{0,T})n_F = (d - 1)F_{0,T} \frac{Y_F}{F_{0,T}} = (d - 1)Y_F.$$

The one-month gross risk-free return $R$ is $1 + r$ (where $r$ is the one-month risk-free rate), with $d < R < u$. To replicate, the futures-bond portfolio positions $Y_F$ and $Y_B$ are chosen so that

$$(u - 1)Y_F + R \cdot Y_B = \text{call}_{T,u,c},$$
$$(d - 1)Y_F + R \cdot Y_B = \text{call}_{T,d,c}.$$

The solution to this system of linear equations is

$$Y_F = \frac{\text{call}_{T,u,c} - \text{call}_{T,d,c}}{u - d},$$
$$Y_B = \frac{1}{R} \left[ \frac{1 - d}{u - d} \text{call}_{T,u,c} + \left( \frac{u - 1}{u - d} \right) \text{call}_{T,d,c} \right].$$

Because a futures contract is worth zero when transacted, the call option value at time 0 is

$$\text{call}_{0,c} = Y_B.$$
2.4. Commodity Conversion Assets as Real Options

The call option is thus valued by financially replicating its date $T$ cash flows.

The ratios $(1 - d)/(u - d)$ and $(u - 1)/(u - d)$ are numbers between 0 and 1 and sum to 1. Thus, they can be interpreted as probabilities. In particular, they are known as the risk-neutral probabilities for the up and down states. Label as $q^{RN}$ the up-state risk-neutral probability. Hence, the time 0 option value can be written as the expected value of the time $T$ option payoffs evaluated using the risk-neutral probabilities $q^{RN} = (1 - d)/(u - d)$ and $1 - q^{RN} = (u - 1)/(u - d)$, discounted at the risk-free discount factor $1/R$:

$$\text{call}_{0,c} = \frac{1}{R} \left[q^{RN} \cdot \text{call}_{T,u,c} + (1 - q^{RN}) \cdot \text{call}_{T,d,c}\right].$$

In words, valuation by replication reduces to the computation of a mathematical expectation discounted using the risk-free discount factor. Because this expectation depends on the risk-neutral, rather than the actual, probabilities of reaching the up or down states, this valuation approach is known as risk-neutral valuation. That is, for valuation (and policy optimization) purposes we can focus on the evolution of the futures price in an alternate risk-neutral world in which no adjustment for risk is required when discounting.

Risk-neutral valuation is a general approach. It is not specific to this simplified setting. This example can be substantially generalized by (i) subdividing the time period between dates 0 and $T$ into small time intervals each of length $\Delta t$; (ii) assuming that over each such time interval the price of the futures for maturity date $T$ can increase by a proportional factor $u$ equal to $\exp(\sigma \sqrt{\Delta t})$ or decrease by a factor $d$ equal to $\exp(-\sigma \sqrt{\Delta t})$, where $\sigma$ is known as the volatility parameter; and (iii) letting the length of these time intervals become smaller and smaller. In the limit, the risk-neutral dynamics of the futures price $F_{t,T}$ converge to Geometric Brownian Motion, with the stochastic differential equation

$$\frac{dF_{t,T}}{F_{t,T}} = \sigma dZ_t,$$

where $dF_{t,T}$ is the instantaneous change in the futures price $F_{t,T}$ and $dZ_t$ is a Standard Brownian Motion increment. The resulting risk-neutral probability distribution of the time $T$ futures price on date


\( T, F_{T,T} \), conditional on the information available on date 0, \( F_{0,T} \), is lognormal with mean \( F_{0,T} \) and variance \( F_{0,T}^2 [\exp(\sigma^2 T) - 1] \).\(^2\) Letting \( \mathbb{E}^{RN}_{0} \) denote risk-neutral expectation and using continuous discounting, the value of the call option on date 0 is

\[
call_{0,c} = e^{-rT} \mathbb{E}^{RN}_{0} \left[ \max \{F_{T,T} - c, 0\} \right],
\]

which, given that the futures price follows a Geometric Brownian Motion, leads to the valuation formula

\[
call_{0,c} = e^{-rT} [F_{0,T} \cdot \Phi(d_1) - c \cdot \Phi(d_2)], \tag{2.2}
\]

where \( \Phi(\cdot) \) is the standard normal cumulative distribution function, \( d_1 \) is \( \ln(F_{0,T}/c) + \sigma^2 T/2]/(\sigma \sqrt{T}) \), and \( d_2 \) is \( d_1 - \sigma \sqrt{T} \).

**Transportation assets and spread options.** Consider a pipeline that transports natural gas from Houston, Texas, to New York City, New York. Suppose that a merchant rents an amount \( Q \) of the capacity of this pipeline for a time period, say one month starting at time \( T \). This merchant can maximize the market value of this monthly block of capacity by optimally trading natural gas on the Houston and New York City spot markets for natural gas. At time \( T \) the merchant optimally purchases an amount of natural gas equal to the rented pipeline capacity on the Houston spot market, ships this natural gas to New York City using this capacity, and sells the shipped natural gas on the New York City spot market if the New York City natural gas price exceeds the Houston natural gas price net of the shipping cost.

Let \( s_{1,T} \) and \( s_{2,T} \) be the spot prices at time \( T \) for the Houston and New York City markets. Denote by \( c \) the marginal shipping cost. The time \( T \) payoff of the optimal shipping policy is

\[
\max \{s_{2,T} - s_{1,T} - c, 0\} \cdot Q. \tag{2.3}
\]

This payoff recognizes that natural gas is simultaneously injected into the pipeline in Houston and delivered in New York City at time \( T \). This simultaneous receipt and delivery of natural gas is realistic because natural gas is moved by displacement and the entire pipeline

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\(^2\)Equivalently, the resulting risk-neutral distribution of \( \ln(F_{T,T}) \) is normal with mean \( \ln(F_{0,T}) - \sigma^2 T/2 \) and variance \( \sigma^2 T \).
from Houston to New York City is filled up with natural gas. It is clear, however, that the delivered natural gas is not physically the same injected natural gas. For simplicity, the payoff (2.3) ignores any fuel used by the pipeline compressors to ship natural gas. It also assumes trading of a monthly block of natural gas, rather than daily trading within the month.

The payoff (2.3) corresponds to the payoff of \( Q \) call spread options on the spot prices \( s_{1,T} \) and \( s_{2,T} \) with strike price \( c \). Different from the payoff (2.1) for the simplified natural gas production asset, the payoff (2.3) involves two distinct commodities: natural gas in Houston at time \( T \) and natural gas in New York City at time \( T \). A contract on natural gas pipeline capacity is thus a cross-commodity conversion asset. It can be interpreted as a real call option on the time \( T \) spread between the natural gas spot prices in Houston and New York City with strike price equal to the marginal shipping cost and notional amount equal to the rented pipeline capacity. The merchant operations of this asset corresponds to the optimal exercise of this option.

Other transportation assets share this spread option feature with natural gas pipelines. Examples include power lines, as well as crude oil tankers and trains that haul coal rail cars. The difference between the latter assets and power lines and natural gas pipelines is that tanker and train payoffs are defined on commodity prices at both multiple location and dates, due to the longer transportation lead time. Transportation assets can also entail multiple sourcing and delivery locations. This aspect can be characterized as a rainbow option, that is, an option to choose the maximum or minimum among multiple prices.

Exchanges such as NYMEX and ICE trade basis swaps for several locations in the United States. These contracts are essentially futures contracts on the difference between the futures price at a given location and the futures price at Henry Hub, the delivery location for the NYMEX natural gas futures contract. The time \( T \) spot prices for Houston and New York City, \( s_{1,T} \) and \( s_{2,T} \), should be identical to the sums of the time \( T \) Henry Hub futures price and the time \( T \) Houston and New York City basis swaps’ prices, respectively, both with time \( T \) maturity. Denote these sums by \( F_{1,T,T} \) and \( F_{2,T,T} \). Thus, the payoff (2.3) is that of a spread option on the difference between the futures
prices $F_{2,t,T}$ and $F_{1,t,T}$ net of the strike price $c$.

Cross-commodity spread options are not directly traded on NYMEX or ICE but they may be traded on OTC markets. However, the time $0 < T$ market value spread $s_{0,c}$ of each one of these options can be determined using risk-neutral valuation, based on a stochastic model of the joint risk-neutral evolution of the prices $F_{1,t,T}$ and $F_{2,t,T}$ during the time interval $[0,T]$:

$$s_{0,c} = e^{-rT} E^{\mathbb{RN}} \left[ \max\{F_{2,t,T} - F_{1,t,T} - c, 0\} \right].$$

For example, suppose that the futures prices $F_{1,t,T}$ and $F_{2,t,T}$ evolve as correlated Geometric Brownian Motions in the risk-neutral world:

$$\frac{dF_{1,t,T}}{F_{1,t,T}} = \sigma_1 dZ_{1,t},$$
$$\frac{dF_{2,t,T}}{F_{2,t,T}} = \sigma_2 dZ_{2,t},$$
$$dZ_{1,t} dZ_{2,t} = \rho dt,$$

where $\sigma_1$ and $\sigma_2$ are the volatilities of the two futures prices, and $dZ_{1,t}$ and $dZ_{2,t}$ are Standard Brownian Motion increments with instantaneous correlation equal to $\rho$. In this case a closed-form formula for the spread option price $s_{0,c}$ is not available, but this price can be approximated as

$$s_{0,c} \approx e^{-rT} \left[ F_{2,0,T} \Phi(d_2) - (F_{1,0,T} + c) \Phi(d_1) \right],$$

where

$$d_2 = \frac{\ln(F_{2,0,T}/(F_{1,0,T} + c)) + \sigma_{1,2}^2 T/2}{\sigma_{1,2} \sqrt{T}},$$
$$d_1 = d_2 - \sigma_{1,2} \sqrt{T},$$
$$\sigma_{1,2} = \sqrt{\sigma_2^2 - 2\rho \sigma_1 \sigma_2 \frac{F_{1,0,T}}{F_{1,0,T} + c} + \left( \frac{\sigma_1}{F_{1,0,T} + c} \right)^2}.$$

Simplified consumption assets and put options. Commodity consumption assets play the opposite role of commodity production assets. Consider a firm that employs a given commodity as its major
input to its manufacturing process. For example, the input commodity could be copper used to manufacture pipes. For simplicity, suppose that the firm can sell its production at time $T$ at the fixed price $K$ (this is not entirely realistic, as the firm could adjust its pricing policy according to the input price, but it represents a situation where output prices are fixed in the short term). This price is net of other (nonrandom) variable manufacturing costs. Suppose that production is known with certainty and is equal to $Q$. Assume, also for simplicity, that the firm does not carry inventory of the input commodity.

The manufacturer’s optimal production policy is to purchase the input commodity, produce, and sell its output at time $T$ if the output price exceeds the input price. The payoff of this policy is

$$\max\{K - s_T, 0\} \cdot Q,$$  \hspace{1cm} (2.5)

The payoff (2.5) is also the payoff of $Q$ European put options on the spot input price at time $T$ with strike price equal to $K$. This observation justifies interpreting the commodity consumption asset as a real put option on the time $T$ spot price of the commodity with strike price equal to the output price and notional equal to the manufacturing capacity. The merchant operations of the manufacturing asset is identical to the optimal exercise of this put option. The time $0 < T$ market value of the manufacturing asset managed as a merchant operations is thus the time 0 value of this futures option. Similar to European call options on commodity futures prices, European put options on such prices are also traded on exchanges (such as LME). The time 0 value of the commodity conversion asset when managed as a merchant operations at time $T$ can thus be observed in the market. It is the value of the futures put option multiplied by the production capacity for time $T$. Moreover, risk-neutral valuation can be applied to value each of these options. Indeed, the time 0 value of a futures put option with expiry on date $T$ and strike price $K$ can be obtained from the time 0 value of the futures call option with the same maturity $T$ and strike price $K$ as

$$\text{put}_{0,K} = \text{call}_{0,K} + e^{-rT}(K - F_{0,T}).$$ \hspace{1cm} (2.6)

These examples illustrate that interpreting commodity conversion assets as real options and the merchant operations of such assets as the
optimal exercise of specific real options is useful because it leads to the maximization of the market values of these assets.

### 2.5 Outline of this Monograph

In some cases, managing a commodity conversion asset as merchant operations is simple, such as in the examples discussed in §2.4. However, for most commodity conversion assets, doing this requires the development of mathematical models that maximize the market value of the asset operating policy based on a stochastic representation of the evolution of commodity and energy prices, subject to certain market pricing consistency criteria. Specifically, this operating policy prescribes how to optimally exercise the managerial flexibility embedded in a commodity conversion asset, that is, the real options that this asset represents, and the market pricing consistency criteria ensure that this policy maximizes the market value of this asset.

The development and use of real option models requires the existence of frictionless financial markets for commodity contracts. Some commodity and energy industries come close to satisfying these requirements, e.g., grains, soybean, natural gas, and oil all have fairly liquid financial markets (e.g., CME, ICE, NYMEX). The real option modeling approach remains a useful approximation even when these requirements are not perfectly satisfied, as it provides a consistent framework for devising operating policies that strive to maximize the market value of an asset. Chapter 3 outlines the theory behind the valuation of commodity options. Chapter 4 illustrates a number of widely used models of the evolution of commodity and energy prices.

The transportation assets discussed in §2.4 are fairly realistic representations of how these assets are operated in practice. In contrast, the commodity production and consumption assets presented in §2.4 simplify many important aspects of how such assets are managed. In particular, these simplified examples neglect a basic aspect that distinguishes most storable commodity conversion assets: inventory. For commodity production assets, inventory represents the reserve of commodity that can be produced and sold over time. For commodity consumption assets, inventory is the input stored in tanks or stockpiles
that is available for use in the manufacturing process. Refining assets feature multiple types of inventory, both for inputs and outputs, but also, possibly, for intermediate products. Thus, for refining, storage is bundled with cross-commodity transformations. Inventory is important because it links decisions across multiple dates and can lead to complex optimal quantity decisions, that is, decisions about a commodity conversion asset operating scale. Optimal merchant operations of commodity storage thus extends the optimal valuation and management of American and Bermudan options with their one-time binary exercise. Switching costs incurred to alter the scale of operations of a commodity conversion asset can also cause intertemporal linkages across decisions.

From this perspective, storage is a foundational element of commodity conversion assets. In a simplified model, a single commodity is purchased from the market at the prevailing spot price, held in stock at a warehouse, and resold back to the spot market at a future date. Figure 2.1 illustrates the essential dynamics of commodity storage. Chapters 5-6 deal with the structure of the optimal policy for commodity storage assets and the benchmarking of policies used to manage such assets in practice, respectively. Chapter 7 shows that the optimal policy structure for commodity storage assets remains relevant for the merchant operations of other assets, including inventory disposal and acquisition assets and swing assets. In contrast, realistic real option modeling of
more complicated commodity conversion assets remains an open area of research and applications. This is one of the trends discussed in Chapter 8.

Figure 2.2 summarizes the merchant operations framework and how the chapters of this monograph relate to the components of this framework. Physical markets include the spot and forward markets where the exchange of commodities and the conversions of these commodities from inputs to outputs occur. Conversion assets are interpreted as real options on the input and/or output commodity prices. Financial markets trade various contracts on these input and output commodities. Physical and financial markets are linked: (i) financial markets provide the information needed to devise operating policies for commodity conversion assets that (strive to) maximize the market-value of these assets, and (ii) the current and future supply and demand conditions in physical markets are reflected in the market prices of the contracts traded in financial markets. Chapter 3 deals with the role played by financial markets in the valuation of the conversion asset physical cash flows. Chapter 4 presents various models of the evolution of spot and futures prices and the impact of supply and demand conditions on price dynamics. Chapters 5-7 focus on operating policies
to manage various commodity conversion assets. Chapter 8 discusses
directions for future research that touch on all the components of mer-
chant operations, including the impact of physical markets on financial
markets.

This monograph takes an integrated finance and operations perspec-
tive on commodity and energy merchant operations. However, Chapters
3-4 are more finance oriented while Chapters 5-7 are more operations
oriented. Those readers who are more interested in finance can focus
on Chapters 3-4 and skim Chapters 5-7, while readers who are more
interested in operations can proceed to Chapters 5-7 after skimming
Chapters 3-4.

2.6 Notes

The categorization of commodity groups and types in §2.1 is from Ge-
man [101, Table 1, p. 13]. Geman [101] discusses in detail the markets
for the commodity groups introduced in §2.1. Schofield [185] provides a
detailed account of commodity markets and derivatives. Clewlow and
Strickland [60], Eydeland and Woyniec [86], Geman [101], Fiorenzani
[93], Burger et al. [40], Pilipovic [171], and Fiorenzani et al. [94] illus-
trate various contracts traded in commodity financial markets, as well
as real options models for valuation and risk management of energy
derivatives and assets. Hull [123, p. 587] is a comprehensive introduc-
tion to derivatives contracts, including futures and options. The col-
lection edited by Ronn [179] includes several contributions on the use
of real options in energy management. Leppard [149] provides a non-
technical introduction to energy derivatives and their role in the risk
management of energy assets.

The real option literature is vast, and no attempt is made here
to provide a comprehensive coverage. Dixit and Pindyck [78] and Tri-
georgis [216] provide introductions to real option concepts. Clewlow
and Strickland [60], Ronn [179], Eydeland and Woyniec [86], Kamin-
ski [130], Geman [101], Hull [123, Chapter 33], and Luenberger [155]
present various commodity and/or energy applications. Smith and Mc-
Cardle [201] discuss practical applications of real option concepts in the
energy industry. Benth et al. [13] is an advanced treatment of energy
price evolution models, a topic discussed in Chapter 4.

The link between the management of commodity conversion assets as merchant operations and the management of real options is implicit in most of this literature. In practice, the merchant operations approach to managing commodity conversion assets has been embraced by commercial banks (see, e.g., Davis [65]) and energy merchants (Ronn [179], Kaminski [130]).

The discussion of the commodity conversion assets considered in §2.4 is based on the work of Brennan and Schwartz [36] on the real option valuation of natural resource production, the research of Deng et al. [74], Eydeland and Wolyniec [86, pp. 59-62], Fleten et al. [95], and Secomandi [189] on the real option valuation of electricity and natural gas transportation infrastructure, and the paper by McDonald and Siegel [160] on the real option valuation of manufacturing firms. Secomandi and Wang [195] consider natural gas transport assets with a network structure.

The example in §2.4 about the valuation of a call option on a futures contract using the binomial model follows Luenberger [155, pages 383-386]. Cox et al. [63] present the binomial model for the valuation of options on a stock and its convergence to the model of Black and Scholes [21], in which the stock price dynamics follow a Geometric Brownian Motion. Formula (2.2) is the Black [20] formula for the price of a call option on a futures contract (see Hull [123, p. 370] for a textbook discussion of this formula). Formula (2.4) is known as Kirk’s approximation for the price of a futures spread option (see Carmona and Durrleman [45]). Expression (2.6) is known as put-call parity (see Hull [123, p. 365]).

Sick [198] discusses the market consistency criteria mentioned at the beginning of §2.5. In particular, the optimal management of commodity and energy real options often gives rise to stochastic dynamic programs whose formulation must satisfy these criteria. Such market consistency criteria are based on assumptions that can be restrictive in applications. Smith [199] points out that even when these assumptions are not fully satisfied, enforcing these criteria provides a consistent approximate framework for informing and supporting managerial decision-making.
This chapter presents a framework for pricing future cash flows paid off by real and financial commodity options. Section 3.1 introduces the idea of risk-neutral (RN) valuation and its connection to option valuation that is consistent with the market prices of related traded securities. Section 3.2 explains valuation in statically complete markets. Section 3.3 describes RN valuation in dynamically complete markets based on Black and Scholes [21] and Black [20]. Section 3.4 explains the role of price-of-risk (POR) assumptions in RN valuation in incomplete markets. Section 3.5 discusses model calibration. Section 3.6 summarizes. Section 3.7 gives pointers to the literature.

3.1 Introduction

Option valuation depends, in general, on the probability beliefs and preferences of investors trading in the market and on the option payoff function. For commodity options, the option payoffs are contingent on commodity prices. The evolution of commodity prices can be represented formally as functions on a probability space \((\Omega, \mathcal{F}, P)\) where \(\Omega\) is the set of all possible states, \(\mathcal{F} = \{\mathcal{F}_t\}\) is a filtration which de-
scribes the possible evolution over time of information about the realized state $\omega \in \Omega$, and $P$ is a probability measure over informational states $\omega_t \in \mathcal{F}_t$. To be more concrete, a state $\omega$ here is a complete description of supply and demand conditions for a commodity at all dates, while an informational state $\omega_t$ can be thought of as what has been publicly reported about commodity market conditions in the *Wall Street Journal* (or the financial press more generally) up through a given date $t$. A probability space like this generalizes the simple binomial tree in Chapter 2 to allow for continuous time, continuous states, and multiple state variables.

Given a probability space, let $c^*_T$ denote an option payoff at a future expiration date $T$ where $c^*_T$ must be measurable with respect to $\mathcal{F}_T$. In other words, $c^*_T$ can only depend on commodity price information that will be known at date $T$. The payoff $c^*_T$ can be determined contractually (as with exchange-traded and OTC call and put options) or operationally (as with conversion assets with embedded real options). Moreover, the operating policy for conversion assets must be both feasible given physical engineering constraints and measurable with respect to the information available over time when decisions are made given the filtration $\mathcal{F}$.

In the absence of arbitrage, any measurable payoff function $c^*_T$ can be valued at date $t < T$ as

$$ c_t = \int_{\omega_T \in \mathcal{F}_T} c^*_T(\omega_T)\varphi(\omega_T|\omega_t)d\omega_T $$

(3.1)

where $\omega_T$ is a possible future state given the information $\mathcal{F}_T$ at date $T$, $\omega_t$ is the state given current information $\mathcal{F}_t$ at date $t$, and $\varphi(\omega_T|\omega_t)$ is a non-negative state price density function giving the market value in state $\omega_t$ at date $t$ of a state-contingent dollar in state $\omega_T$ at date $T$. The state prices incorporate everything that matters at date $t$ for valuing future $\omega_T$-contingent cash. In particular, state prices depend on the time-value-of-money between dates $t$ and $T$, the market’s beliefs about the probability of state $\omega_T$ occurring at $T$ given that $\omega_t$ is the

---

1 See Harrison and Kreps [116] and Duffie [83] Chapters 1 and 6 for more on multi-period asset pricing and state prices. See Delbaen and Schachermayer [72] for a technical definition of the absence of arbitrage.
3.1. Introduction

current state at \( t \), and the market’s preferences in state \( \omega_t \) for state-
contingent cash in the future state \( \omega_T \). For example, the market may
have a stronger preference for insurance in some futures states (e.g.,
bad macroeconomic states) than in others (e.g., good macroeconomic
states).\(^2\) If the risk-free interest rate is a constant \( r \), then the state price
valuation equation (3.1) can be manipulated as

\[
c_t = \left[ \int_{\omega_T \in \mathcal{F}_T} c^*_T(\omega_T) \frac{\varphi(\omega_T|\omega_t)}{\int_{\omega_T \in \mathcal{F}_T} \varphi(\omega_T|\omega_t)d\omega_T} d\omega_T \right] \left[ \int_{\omega_T \in \mathcal{F}_T} \varphi(\omega_T|\omega_t)d\omega_T \right]
\]

\[
= \left[ \int_{\omega_T \in \mathcal{F}_T} c^*_T(\omega_T) q^{RN}(\omega_T|\omega_t)d\omega_T \right] e^{-r(T-t)}
\]

(3.2)

to obtain the risk-neutral (RN) valuation equation:\(^3\)

\[
c_t = \mathbb{E}_t^{RN}[c^*_T]e^{-r(T-t)}
\]

(3.3)

where \( \mathbb{E}_t^{RN} \) denotes RN expectation given date \( t \) information. The second equality in (3.2) follows from (i) the fact that the integral of the state prices \( \int_{\omega_T \in \mathcal{F}_T} \varphi(\omega_T|\omega_t)d\omega_T \) equals the price of a risk-free discount
bond \( e^{-r(T-t)} \) (i.e., of a riskless payoff of $1 at \( T \) in each possible
state \( \omega_T \)) and (ii) an interpretation of scaled state prices as preference-
adjusted risk-neutral probabilities \( q^{RN}(\omega_T|\omega_t) \). The RN probabilities
are non-negative and integrate to one but can differ from the objective
probabilities because the RN probabilities are constructed from state
prices which depend on market preferences as well as on objective
probabilities.\(^4\)

The RN valuation equation (3.3) has important theoretical and
practical implications. One immediate implication is that all of the
properties of option prices and their risk sensitivities follow directly

\(^2\) The well-known Capital Asset Pricing Model (CAPM) predicts that risk-averse investors
value cash in down stock-market states (when macroeconomic conditions tend to be bad)
more than in up stock-market states because their wealth is positively correlated with the
market return. See Luenberger [??] Chapter 7 or any investments textbook.

\(^3\) See Schwartz [186], Miltersen and Schwartz [162] and Casassus and Collin-Dufresne [48]
for models of commodity option pricing with stochastic interest rates.

\(^4\) Sometimes RN probabilities are decomposed as \( q^{RN}(\omega_T|\omega_t) = p(\omega_T|\omega_t)m(\omega_T|\omega_t) \) where
\( p(\omega_T|\omega_t) \) is the objective probability and \( m(\omega_T|\omega_t) \) is called the pricing kernel and
represents the equilibrium pricing impact of investor preferences.
from the option payoff formula and the RN dynamics of the state variables driving the commodity price filtration. Another implication of (3.3) is that the annualized RN expected return on any claim on future state-contingent cash flows—that is to say, on any investible asset—is the risk-free interest rate $r$. However, the RN valuation representation of asset prices imposes no restrictions on the RN drift of variables which are not prices of investible assets. This point will be important in our discussion of the RN dynamics of non-asset commodity prices (e.g., commodities that are perishable or which are not practical stores of value like electricity or some agricultural goods) as well as for the RN dynamics of variables like weather and stochastic volatility factors.

A practical use of RN valuation is that equation (3.3) provides a tractable numerical procedure for computing the value of options when analytic formulas for option prices are not available. First, $N$ realizations of the underlying state $\omega_T$ are simulated under the RN probability measure using Monte Carlo. Second, option payoffs $c_T^\ell$ are computed for each simulated realization $\ell = 1, \ldots, N$ of the state $\omega_T(\ell)$, averaged, and discounted at the risk-free interest rate to get an estimated option valuation:

$$\hat{c}_t = \frac{\sum_{\ell=1}^{N} c_T^\ell}{N} \cdot e^{-r(T-t)}.$$  

Since averages are unbiased estimators of expected values, the Monte Carlo estimate $\hat{c}_t$ is an unbiased estimate of the option price $c_t$. The Monte Carlo implementation of RN valuation is a workhorse tool used in practice to value gas storage, power plants, and other complex commodity conversion assets.

Real options often pay cash flows at not just a single date but rather a stream of cash flows $c_1^s, c_2^s, \ldots, c_M^s$ at a sequence of dates $T_1, T_2, \ldots$.

\[ \text{Risk sensitivities to the state variables which define the state } \omega_t \text{ and to model parameters are known as an option's Greeks (e.g., delta, gamma, vega). See Hull [123] Chapter 18.} \]

\[ \text{The return over a time interval } \Delta \text{ on any claim on state-contingent cash is } [c_{t+\Delta} - c_t] / c_t = [E^{\text{RN}}_{t+\Delta}[c_T^s] e^{-r(T-t-\Delta)} - E^{\text{RN}}[c_T^s] e^{-r(T-t)}] / E^{\text{RN}}[c_T^s] e^{-r(T-t)}. \] Taking RN expectations with respect to date $t$ information, using iterated expectations, and then taking the limit as $\Delta t \to 0$ gives $E^{\text{RN}}[dc_t / c_t] = r \, dt$. 

\[ \text{5 Risk sensitivities to the state variables which define the state } \omega_t \text{ and to model parameters are known as an option's Greeks (e.g., delta, gamma, vega). See Hull [123] Chapter 18.} \]

\[ \text{6 The return over a time interval } \Delta \text{ on any claim on state-contingent cash is } [c_{t+\Delta} - c_t] / c_t = [E^{\text{RN}}_{t+\Delta}[c_T^s] e^{-r(T-t-\Delta)} - E^{\text{RN}}[c_T^s] e^{-r(T-t)}] / E^{\text{RN}}[c_T^s] e^{-r(T-t)}. \] Taking RN expectations with respect to date $t$ information, using iterated expectations, and then taking the limit as $\Delta t \to 0$ gives $E^{\text{RN}}[dc_t / c_t] = r \, dt$. 

3.1. Introduction

Using RN valuation, the value of a stream of cash flows is

\[ c_t = \sum_{i=1}^{M} E_t^{RN}(c_{T_i}^*) e^{-r(T_i-t)}. \]  

(3.5)

The stream of cash flows for a conversion asset is — as explained in Chapter 2 — a stream of operating cash flows over time. For example, mines generate streams of net profits over time tied to mineral prices. Electric power plants produce streams of operating profits tied to the spark spread between power and fuel prices. Natural gas storage – which is considered in detail in Chapters 5 and 6 – yields a stream of negative cash flows (in months in which gas is purchased and stored) and positive cash flows (in months in which stored gas is withdrawn and sold). Moreover, real option cash flows are often dynamically linked over time if operating decisions at one date affect the distribution of cash flows at later dates. Inventory, ramp-up/cool-down requirements, and start-up/shut-down costs are examples of such dynamic interactions.

The purpose of an option pricing model is to specify the RN probabilities with which to value future state-contingent cash flows. Since the existence of state prices – and, hence, of RN probabilities – is ensured by absence of arbitrage, the focus in this chapter is on two related issues: When are RN probabilities unique? And how are RN probabilities identified? In particular, this chapter reviews three approaches to identify RN probabilities. The first approach is possible in statically complete markets. The second is possible in dynamically complete markets. The third requires so-called price-of-risk assumptions when the market is incomplete. While the discussion here is exposited in terms of commodities, the uniqueness and identification of the RN dynamics of an option’s underlying variable are generic issues. They arise with options on any investible asset (e.g., stocks, bonds) and on underlyings which are not directly investible (e.g., interest rates, weather, volatility, default events).

A key consideration in option pricing is what exactly constitute a “state.” In its most complete form, an informational state \( \omega_t \in \mathcal{F}_t \) in a commodity option pricing model includes the current spot price \( s_t \) and also any other factors (i.e., other random variables like stochastic volatility) that affect the future dynamics of spot prices plus the prior
history of all state variables. For example, weather is a natural state variable for electricity prices, and international political uncertainty might be a state variable for oil price volatility. In this chapter, the focus is on state variable dynamics and option pricing from a fairly generic vantage point. Chapter 4 then describes a variety of different commodity option pricing models in terms of their specific representations of states, price dynamics, and RN probabilities.

### 3.2 Statically Complete Markets

Traded securities are bundles of state-contingent cash flows in various future dates and states. A market is said to be *statically complete* if any combination of future state-contingent payoffs can be constructed using buy-and-hold positions in market-traded securities. More formally, a market is statically complete if there are as many traded securities with linearly independent cash flows as there are states.

In a statically complete market, state prices can be recovered from the market prices of traded securities. Let \( t_0 \) denote the current date and let \( t_1, \ldots, t_M \) denote discrete future payment dates. Let \( K_j \) denote the total number of discrete (for simplicity) possible states at future payment date \( t_j \). The total number of future date-state pairs is \( K = \sum_{j=1}^{M} K_j \). Let \( \varphi_k \) denote the state price at date \( t_0 \) of a state-contingent dollar in the \( k \)-th possible future state-date pair. Let \( \text{CF}_{i,k} \) denote the future cash flow paid off by a particular traded security \( i \) in the \( k \)-th possible future date-state, and let \( P_{i,t_0} \) denote the price of security \( i \) on date \( t_0 \). State prices equate the values of the future security cash flows with the current date \( t_0 \) security prices. If we have \( N \) traded securities with linearly independent future cash flows, then the market prices of these securities imply the system of linear equations

\[
\begin{align*}
P_{1,t_0} &= \varphi_1 \cdot \text{CF}_{1,1} + \varphi_2 \cdot \text{CF}_{1,2} + \cdots + \varphi_K \cdot \text{CF}_{1,K}, \\
P_{2,t_0} &= \varphi_1 \cdot \text{CF}_{2,1} + \varphi_2 \cdot \text{CF}_{2,2} + \cdots + \varphi_K \cdot \text{CF}_{2,K}, \\
&\quad \cdots \\
P_{N,t_0} &= \varphi_1 \cdot \text{CF}_{N,1} + \varphi_2 \cdot \text{CF}_{N,2} + \cdots + \varphi_K \cdot \text{CF}_{N,K}.
\end{align*}
\]

The solution for the \( \varphi_k \)s in (3.6) is unique if the number of traded securities with linearly independent future cash flows, \( N \), equals the
number of future dates-states $K$. However, in practice, there are typically many more future dates-states than traded securities, so markets are rarely statically complete. Hence, state prices cannot be uniquely identified algebraically from an under-determined system of $N$ equations in $K > N$ unknowns. The problem becomes even more severe with continuous-states (as in (3.1)) when there is an infinite number of possible states but only a finite number of traded securities.

State prices can still exist in incomplete markets, but they are not uniquely identified by traded security prices. In the absence of arbitrage, there are multiple possible state price vectors which are each consistent with the traded security prices in the market. Option pricing models impose additional structure on state prices in order to determine state prices in an incomplete market. An option pricing model is correct if its assumed additional structure on state prices is consistent with the structure in fact imposed on state prices by the actual economics of the market.

3.3 Dynamically Complete Markets

Markets which are statically incomplete may still be dynamically complete if there are dynamic trading strategies with traded securities which can replicate any state-contingent payoff. In particular, a dynamic trading strategy is a rule for buying and selling securities over time. The link between dynamic completeness and option pricing was first recognized in Black and Scholes [21]. There are two common approaches to commodity option pricing in dynamically complete markets. The first assumes that the underlying commodity is itself a traded asset. This approach applies to commodities like gold which are investible stores of value over time. The second approach, due to Black [20], assumes that a traded commodity futures contract is available. This second approach can be used to price a restricted set of option payoffs when the underlying physical commodity itself is not an investible asset. In the context of physical conversion assets, for example, it can be used to price real options on electricity (e.g., power plants) when electricity futures are traded but electricity itself is not directly investible.
Most of the option pricing models used in practice have continuous
time and continuous states. This raises the mathematical bar. Unlike
the binomial tree in Chapter 2 and the finite $K$ states in §3.2, stochastic
calculus is used to model the underlying variable dynamics in terms of
stochastic differential equations.

**Commodities which are investible assets.** The Black-Scholes ap-
proach to identify RN dynamics follows from two key assumptions.
First, the underlying commodity is assumed to be an investible asset
(e.g., like gold). Second, the objective dynamics of the spot price of the
commodity $s_t$ are assumed to follow a general Markov Ito process:

$$ds_t = \mu(s_t,t)s_t \, dt + v(s_t,t)s_t \, dZ_t$$

(3.7)

where $dZ_t$ is a Standard Brownian Motion. These Ito dynamics gen-
eralize the original Black and Scholes [21] derivation for Geometric
Brownian Motion (GBM) where the local drifts and volatilities are
constants $\mu(s_t,t) = \mu$ and $v(s_t,t) = \sigma$. The assumption of an univari-
ate Markov process for spot prices means that the spot price $s_t$ itself is
the only state variable needed to describe commodity price dynamics.
In particular, there are no additional random factors (e.g., weather) af-
fecting its drift and volatility. Given these two assumptions, generalized
Black-Scholes RN dynamics can be derived in two steps.

The first step is the derivation of the Black-Scholes partial differ-
tential equation. Over time only two things change in a Black-Scholes
economy – the spot price $s_t$ and the date $t$. As a result, option prices are
functions $c(s_t,t)$ of just two state variables, $s_t$ and $t$, where everything
else at date $t$ is parametrically fixed by assumption (e.g., interest rates,
the forms of the local drift and volatility functions $\mu(\cdot, \cdot)$ and $v(\cdot, \cdot)$),
contractually fixed (e.g., option strike prices and expiration dates), and
historically fixed (e.g., the path of prior spot prices before date $t$ when
valuing path-dependent options). The valuation function $c(s_t,t)$ must,
however, be consistent with absence of arbitrage. This no-arbitrage

---

7For simplicity, we just consider Markov Ito processes, but these dynamics can be general-
zeed to incorporate path-dependence in the drifts and volatilities. See Shreve [197] Chapter
4 for more on Ito processes.
condition imposes restrictions on the types of functions $c$ which value commodity price-contingent future cash flows.

Consider a portfolio consisting of a long position in a given option combined with a position consisting of $-\delta$ units of the underlying commodity. In particular, this is only possible if the physical commodity is a non-perishable, storable, and, hence, investible asset that can be included in an investment portfolio. At date $t$ this portfolio is worth $c_t - \delta s_t$. Since $c_t$ is a function $c(s_t, t)$, Ito's Lemma tells us how this portfolio's value changes in response to the passage of time $dt$ and changes $ds_t$ in the underlying commodity spot price:

$$dc_t - \delta ds_t = \left[ \left( \frac{\partial c_t}{\partial t} + \frac{1}{2} v^2(s_t, t) s_t^2 \frac{\partial^2 c_t}{\partial s^2} \right) dt + \frac{\partial c_t}{\partial s} ds_t \right] - \delta ds_t \quad (3.8)$$

provided that the partial derivatives $\partial c_t/\partial t$, $\partial c_t/\partial s$, and $\partial^2 c_t/\partial s^2$ exist and are continuous.

The portfolio value dynamics have two components: A deterministic part due to the nonrandom passage of time $[\partial c_t/\partial t + (1/2) v^2(s_t, t) s_t^2 \partial^2 c_t/\partial s^2] dt$ and a risky part due to random changes in the spot price $(\partial c_t/\partial s - \delta) ds_t$. Setting the commodity position $-\delta = -\partial c_t/\partial s$ makes the combined option+commodity portfolio riskless. In particular, the implicit exposure to spot price risk through the option, $\partial c_t/\partial s$, is offset by the direct exposure from the $-\partial c_t/\partial s$ commodity position. To avoid arbitrage between this riskless option+commodity portfolio and riskless bonds, the function $c(s_t, t)$ must satisfy

$$dc_t - \frac{\partial c_t}{\partial s} ds_t = \left( c_t - \frac{\partial c_t}{\partial s} s_t \right) r dt. \quad (3.9)$$

In words, the change in the value of the hedged option+commodity portfolio must be the same as would be earned if an equal amount of money, $c_t - (\partial c_t/\partial s)s_t$, were instead invested at the risk-free interest rate. This leads to the Black-Scholes partial differential equation (PDE):

$$\frac{\partial c_t}{\partial t} + \frac{1}{2} v^2(s_t, t) s_t^2 \frac{\partial^2 c_t}{\partial s^2} = \left( c_t - \frac{\partial c_t}{\partial s} s_t \right) r. \quad (3.10)$$

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8 For an explanation of Ito's Lemma, see Shreve [197] Chapter 4 or Hull [123] Chapter 13.
This equation has infinitely many solutions $c$ corresponding to different arbitrage-free valuation functions. To pick out the specific function which prices a given option, a boundary condition is imposed on (3.10) requiring that the function $c_t$ converge to the payoff function $c^*_T$ on the payoff date $T$.

In the special case of a GBM, for which the local volatility is constant (i.e., $\nu(s, t) = \sigma$), the Black-Scholes PDE can be explicitly solved to value a variety of options (including European calls and puts) in closed-form. However, our interest here is not in closed-form solutions per se, but rather to identify RN commodity price dynamics as an input into RN valuation for a large class of heteroskedastic processes and complex operational payoffs.

As an aside, note that option payoffs can be exactly replicated, under the Black-Scholes assumptions, by a self-financing dynamic trading strategy. Rearranging (3.9) gives

$$dc_t = \frac{\partial c_t}{\partial s} ds_t + \left( c_t - \frac{\partial c_t}{\partial s} s_t \right) r dt$$

which says that the dollar price change in an option $dc_t$ over each instant $dt$ is exactly mirrored by the dollar change in the value of a portfolio consisting of $\partial c_t / \partial s$ units of the commodity and $c_t - (\partial c_t / \partial s)s_t$ dollars in riskless Treasury bills.\(^9\) This argument is the essential intuition for showing that the Black-Scholes market is dynamically complete.\(^10\) A discretized version of (3.11) is often used in practice as an approximate hedge for option positions. In particular, taking an offsetting position of $\partial c_t / \partial s$ units of the underlying investible commodity to hedge commodity price risk in an option position – with or without t-bills – is known as delta hedging.

\(^9\)The cost of the replicating commodity+bond portfolio on day $t$ is the price of the option $\partial c_t / \partial s s_t + c_t - \partial c_t / \partial s s_t = c_t$. Thus, the replicating trading strategy is “self-financing” through time because the change in the value of the portfolio exactly matches the change in the cost of the replicating portfolio instant by instant.

\(^10\)As presented, the argument here, based on Ito’s Lemma, only applies to option payoff functions $c^*_T$ which are limits of functions $c(s_t, t)$ with continuous partial derivatives $\partial c_t / \partial t$, $\partial c_t / \partial s$, and $\partial^2 c_t / \partial s^2$ as $t \to T$. See Shreve [197] Chapter 5 for a proof that a replicating strategy exists in general for any measurable payoff function $c^*_T$ of $\omega_T$. 
modety price dynamics uses the Black-Scholes PDE to identify a set of so-called equivalent economies and then makes a convenient choice of a particular equivalent economy in which valuation is tractable. In the Black-Scholes model, an economy is described by a collection \( \{U, \mu, v, r, s_0\} \) consisting of investor preferences \( U \) (i.e., investor utility functions), an Ito process (3.7) for the objective underlying commodity price dynamics, a risk-free interest rate \( r \), and an initial commodity spot price \( s_0 \) at a beginning date \( t_0 \). We know that option prices on date \( t_0 \) depend on \( v(\cdot, \cdot), r, \) and \( s_0 \) since they appear explicitly in the Black-Scholes PDE – together with the strike price, expiration date, and any other contractual terms in the payoff \( c^*_T \) in the associated boundary condition – but option prices do not depend directly on either the underlying commodity’s drift \( \mu(\cdot, \cdot) \) or on investor risk preferences \( U \).

Because the Black-Scholes model values options based on a risk-free arbitrage argument, neither \( \mu \) nor \( U \) affect the arbitrage-free option pricing functions \( c \) implicitly described by the Black-Scholes PDE.

This last observation leads to a key insight: Option prices in the true economy \( \{U, \mu, v, r, s_0\} \) are unchanged in any alternate (i.e., imaginary) economy \( \{U_{alt}, \mu_{alt}, v, r, s_0\} \) with different investor preferences \( U_{alt} \) and drifts \( \mu_{alt}(\cdot, \cdot) \) but where the local volatility function \( v(\cdot, \cdot) \), interest rate \( r \), and initial spot price \( s_0 \) are unchanged. This follows since both economies have the same Black-Scholes PDE. Accordingly, we can define a set of equivalent economies with different preferences \( U_{alt} \) and drifts \( \mu_{alt}(\cdot, \cdot) \) in which all option prices are the same as in the true economy.

Options can be priced numerically either by arbitrage (i.e., evaluating the replicating portfolio by solving the PDE) or, alternatively, by Discounted Cash Flows (DCF) valuation. In the actual true economy, an option’s DCF valuation is\(^\text{11}\)

\[
c_t = \frac{E_t[c^*_T]}{1 + \text{RAD}_t}
\]

\(^\text{11}\)In general, different options have different discount rates in the true economy since they have different risk exposures. The DCF formula is written here using a simple discount rate, rather than a continuously compounded rate, for ease of comparison with the usual textbook DCF formulation. For more on DCF valuation, see any introductory finance textbook.
where $E_t$ denotes the objective expectation given the true stochastic process for the underlying commodity and $\text{RAD}_t$ is the appropriate risk-adjusted discount rate specific for a particular given option at date $t$. Similarly, in a particular equivalent economy, this same valuation can also be computed as

$$c_t = \frac{[E_t^{alt}[c_T^{*}]]}{1 + \text{RAD}_{alt}^t}$$

(3.13)

where $E_t^{alt}$ denotes expectations given the alternative Ito process with drift $\mu_{alt}$ and $\text{RAD}_{alt}^t$ is the option’s corresponding alternate risk-adjusted discount rate. In other words, DCF in an equivalent economy, with the appropriate alternative expectation and alternative discount rate, gives the same option valuation as DCF valuation in the true economy. The difficulty with DCF valuation in the true economy is that the true discount rate, $\text{RAD}_t$, for most options is usually not known a priori. One way to solve the unknown true discount rate problem is to search for a convenient equivalent economy in which the option’s risk-adjusted discount rate, $\text{RAD}_{alt}^t$, is known a priori. That is to say, we pick alternate preferences $U^{alt}$ for which we know how to discount in the chosen equivalent economy. Since the Black-Scholes PDE still holds in this chosen equivalent economy, option prices computed there by DCF valuation are, by construction, *identical* to option prices in the true economy.

A convenient choice of an equivalent economy is a risk-neutral economy $\{U^{RN}, \mu^{RN}, v, r, s_0\}$. With risk neutrality, all expected future cash flows (including those of risky options) are just discounted for time-value-of-money at the risk-free interest rate, but not discounted for risk. Moreover, the risk-free rate in the equivalent RN economy is the same as in the actual economy by the definition of equivalent economies. For the equivalent RN spot commodity market to clear (i.e., for supply to equal demand), the RN expected return on investible commodities must equal the risk-free rate, $\mu^{RN} = r$. If not, the commodity and bond markets would not clear at the spot price $s_0$ and interest rate $r$ because there would be infinite demand from risk-neutral investors either to buy the commodity and borrow (if $\mu^{RN} > r$) or to short the commodity and lend (if $\mu^{RN} < r$).
To summarize, DCF valuation in the equivalent RN economy implies that the price of a commodity derivative with a payoff $c_T^*$ is

$$c_t = c_t^{RN} = E_t^{RN}[c_T^*]e^{-r(T-t)}$$  \hspace{1cm} (3.14)

where the underlying spot commodity price follows the equivalent RN process

$$ds_t = rs_t dt + v(s_t, t) s_t dZ_t.$$  \hspace{1cm} (3.15)

The first equality in (3.14) says that option prices in the true economy are the same as option prices in the equivalent RN economy (by definition of an equivalent economy), and the second equality is DCF valuation in the equivalent RN economy. Although our derivation here uses Ito’s Lemma — which implicitly places restrictions on the payoff function $c_T^*$ to be priced via the differentiability conditions on the valuation function $c_t$ — Shreve [197] Chapter 5 has a general proof of RN valuation that does not use Ito’s Lemma and, thus, that applies for any payoff $c_T^*$ that is measurable with respect to date $T$ commodity price information.

The incremental content in Black-Scholes relative to generic RN Valuation in (3.3) is that (3.15) gives explicit RN dynamics for the spot price state variable over which RN expectations are taken. We call this approach the generalized Black-Scholes model since it extends the original Black-Scholes model to allow for heteroskedasticity. In the special case in which the underlying commodity prices follow a GBM, the valuation in (3.14) can be computed in closed-form for European calls, puts, and many other options (see Hull [123] Chapter 25). Otherwise, option prices can be computed numerically as in (3.4).

Later in this chapter we shall see that so-called price-of-risk (POR) assumptions are needed to identify the RN spot price dynamics in dynamically incomplete markets. However, no such POR assumption is needed in the RN identification in a dynamically complete Black-Scholes market. That is part of the appeal of the Black-Scholes approach. However, many commodities – electricity or natural gas during peak demand times – are not investible assets and, thus, the riskless hedge step in the Black-Scholes identification of the RN dynamics is not possible. As will be seen in Chapter 4, the non-asset nature of such
Commodity option valuation makes the properties of the RN Black-Scholes spot price process (3.15) inappropriate for valuing options on non-asset commodities.

**Commodity futures prices as the underlying variable.** A futures contract with delivery at date τ specifies a futures price which the long investor in the futures contract pays to the short investor to buy a unit of a commodity. No money changes hands at initiation, so the futures price $F_{t,\tau}$ at each date $t$ is set to make the date $t$ value of a futures contract with delivery at date $\tau$ equal to zero. In particular, the futures price is not the value of the futures contract; rather it is a contractual term which the market determines so as to set the value of the futures contract to zero. Over the life of a futures contract, the futures prices on old futures contracts with delivery at $\tau$ are reset to the new futures prices with compensatory payments between the long and short sides equal to the futures price change.12

Black [20] develops a riskless hedge argument for pricing options whose terminal payoff $c_T$ at an expiration date $T$ is measurable with respect to information generated by the futures price $F_{t,\tau}$ for a specific delivery date $\tau$.13 In other words, the payoff is measurable with respect to a filtration induced by the evolution of the single futures price $F_{t,\tau}$ by itself. In general, this single-futures price filtration is less informative that the full commodity spot price filtration. A “state” in the Black model is less informative than a state in Black-Scholes because, without additional assumptions, the dynamics of the futures price for one maturity does not necessarily imply anything about the dynamics of futures prices for other maturities. Thus, we refer to this valuation approach as the *single-futures Black model* to emphasize this important limitation. The single-futures limitation will be relaxed later in §4.3 in Chapter 4 when we discuss derivative valuation using the dynamics of the full term structure of futures prices.

The Black model does not require that the underlying commodity is

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12 In practice, this settlement/payment process happens daily, but, in some models, it is assumed to happen continuously.

13 A special case is when the option expiration date $T$ and the underlying futures maturity date $\tau$ are the same.
3.3. Dynamically Complete Markets

an investible asset – i.e., the commodity can be non-storable/perishable such as electricity – but it does assume that the futures contract with date $\tau$ delivery is traded and that the objective dynamics of the futures price for date $\tau$ delivery are a Markov Ito process:

$$dF_{t,\tau} = \mu(F_{t,\tau}, t)F_{t,\tau} dt + v(F_{t,\tau}, t)F_{t,\tau} dZ_t. \quad (3.16)$$

These dynamics generalize Black [20] which originally assumed the futures price follows a Geometric Brownian Motion. In the Black model, the futures price $F_{t,\tau}$ is the only state variable needed, along with time $t$, to describe its own dynamics. Hence, the Black valuation at date $t$ for an option whose payoff $c^*_T$ is only contingent on futures prices associated with delivery date $\tau$ is a function $c(F_{t,\tau}, t)$ of the current futures price $F_{t,\tau}$ and time $t$ with everything else again parametrically, contractually, or historically fixed. To find valuation functions $c$ which are consistent with the absence of arbitrage, consider a portfolio consisting of a long position in one option combined with a position $-\delta$ in the date-$\tau$ delivery futures contract. Since futures prices are set on date $t$ to make futures contracts have zero market value, this portfolio at date $t$ is worth $c_t - \delta 0 = c_t$. Ito’s Lemma and a variation on the Black-Scholes riskless hedge logic imply that, to avoid arbitrage, the Black PDE must hold:

$$\frac{\partial c_t}{\partial t} + \frac{1}{2}v^2(F_{t,\tau}, t)F_{t,\tau}^2 \frac{\partial^2 c_t}{\partial F^2} = c_t r. \quad (3.17)$$

In the Black PDE, there is no analogue to the $(\partial c_t/\partial s)_t$ term in the Black-Scholes PDE since futures contracts, by definition, have a market value of zero at each date $t$.\(^{14}\)

Continuing with the RN identification argument, the Black PDE identifies a set of equivalent economies $\{U^{alt}, \mu^{alt}, v, r, F_{t,\tau}\}$ with potentially different investor preferences $U^{alt}$ and futures price drift $\mu^{alt}(F_{t,\tau}, t)$ than in the actual true economy. Once again, a convenient choice of an equivalent economy is a RN economy $\{U^{RN}, \mu^{RN}, v, r, F_{t,\tau}\}$ so that expected option cash flows are just discounted for time-value-of-money at the risk-free interest rate. Since

\(^{14}\)The replicating equation with futures which is analogous to (3.11) is $dc_t = (\partial c_t/\partial F)dF_{t,\tau} + c_trdt$. This says that the change in the option price is replicated by a position $\partial c_t/\partial F$ in the futures contract and $c_t$ dollars in t-bills.
futures have a zero up-front investment cost, the RN expected futures price drift $\mu^{\text{RN}}$ must be zero for markets to clear at the futures price $F_{t_0,\tau}$. Otherwise, risk-neutral investors would try to take unboundedly large long or short futures positions and markets in the equivalent RN economy would not clear at the current futures price $F_{t_0,\tau}$ (which would contradict part of the definition of an equivalent RN economy).

To summarize the key points about the generalized Black model, commodity futures derivatives are priced as discounted RN expectations

$$c_t = \mathbb{E}^{\text{RN}}_t [c_T] e^{-r(T-t)}$$

where the underlying RN futures price dynamics are

$$dF_{t,\tau} = v(F_{t,\tau}, t) F_{t,\tau} dZ_t.$$  

The well-known Black futures call price (2.2) in Chapter 2 is a special case of this formula when the underlying futures price follows a GBM.

Although similar to Black-Scholes, a significant limitation of the single-futures Black model is that it can only price options with payoffs $c_T^\tau$ which are functions measurable with respect to the sub-filtration induced by the particular underlying futures price $F_{t,\tau}$ with a particular delivery date $\tau$ whose RN dynamics are given in (3.19). This is in contrast to Black-Scholes which can price any option payoff $c_T^\tau$ that is measurable with respect to any aspect of the full commodity price process filtration. The single-futures Black model cannot be used – without additional assumptions – to price options on spot prices at dates other than date $\tau$ (at which time the instantaneous futures price $F_{\tau,\tau}$ equals the spot price $s_\tau$) or to price options whose payoffs depend on futures prices $F_{t,\tau'}$ for other delivery dates $\tau' \neq \tau$. To extend the single-futures Black approach to price general payoffs measurable with respect to commodity spot prices at any date, or futures prices for arbitrary delivery dates, requires a model of the dynamics of the entire term structure of futures prices (see §4.3 in Chapter 4).

**Using a commodity option to identify RN dynamics.** The reason for the limitations on option pricing in the single-futures Black approach is that the connection from the single futures price (whose RN dynamics $dF_{t,T}$ are known a priori) back to the more primitive state
space induced by the underlying RN commodity spot price dynamics is unknown. In other words, the futures price function relating the futures price \( F_{t, \tau} \) and commodity spot prices \( s_t \) is not known. However, if this relation is known a priori and is monotone, then we can again uniquely identify the RN commodity price dynamics. Moreover, this point is not limited just to futures. It applies to all derivatives subject to a natural monotonicity condition. To see this, suppose that the objective dynamics of the underlying commodity spot price – where the commodity is not an investible asset – are known to be
\[
d s_t = m(s_t, t)dt + v(s_t, t)dZ_t, \tag{3.20}
\]
and that there is an option with expiry \( T \) with a known pricing function \( c(s_t, t) \) relating the option valuation and the spot price. From Ito’s lemma, the dynamics of the known option price are
\[
d c_t = \left[ \frac{\partial c_t}{\partial t} + \frac{1}{2} v^2(s_t, t) \frac{\partial^2 c_t}{\partial s^2} + \frac{\partial c_t}{\partial s} m(s_t, t) \right] dt + \frac{\partial c_t}{\partial s} v(s_t, t) dZ_t. \tag{3.21}
\]
Given these assumptions, the question is: What are the RN dynamics
\[
d s_t = m^{RN}(s_t, t)dt + v^{RN}(s_t, t)dZ_t \tag{3.22}
\]
for commodity spot prices? Since the quadratic variation of an Ito process is unchanged under the RN measure, we have \( v^{RN}(s_t, t) = v(s_t, t) \).\(^{15}\) Turning to the RN drift, since options are assets, the known option’s RN expected return is the risk-free interest rate, which implies
\[
\frac{\partial c_t}{\partial t} + \frac{1}{2} v^2(s_t, t) \frac{\partial^2 c_t}{\partial s^2} + \frac{\partial c_t}{\partial s} m^{RN}(s_t, t) = r c(s_t, t). \tag{3.23}
\]
Rearranging this gives
\[
m^{RN}(s_t, t) = r c(s_t, t) - \frac{\partial c_t}{\partial t} - (1/2) v^2(s_t, t) \frac{\partial^2 c_t}{\partial s^2} \frac{\partial c_t}{\partial s} \tag{3.24}
\]
where, since the function \( c(s_t, t) \) is known, all of the partial derivatives on the right-hand side of (3.24) are known, and, thus, the RN commodity price drift is determined and well-defined provided that \( \frac{\partial c_t}{\partial s} \neq 0 \).

\(^{15}\)See Shreve [197], Chapter 3.
Since Ito’s Lemma requires that $\frac{\partial c_t}{\partial s}$ be continuous, the requirement $\frac{\partial c_t}{\partial s} \neq 0$ means that the option price function is monotone in the commodity spot price. Given the RN spot price dynamics, we can now use numerical RN valuation, as in equation (3.4), to value any commodity-linked cash flows that are measurable with respect to the commodity price process. In addition, if the known option can be traded in a liquid market, it can be used to hedge spot-price-induced randomness $ds_t$ in other commodity-linked derivatives.

Unfortunately, it is rare that the option valuation function $c$ is known a priori for even one option. As a result, this approach and the RN drift identification in (3.24) is typically not of practical use. However, this discussion does illustrate some of the subtleties in the connection between spot prices and derivative prices when identifying RN spot price dynamics.

### 3.4 Dynamically Incomplete Markets

Markets are dynamically incomplete when the underlying variable for an option is driven by Brownian Motions, jumps, or auxiliary state variables which are not investible. For example, suppose that the spot prices for a commodity follow a jump-diffusion process:

$$ds_t = m(x_t, s_t, t)dt + v(x_t, s_t, t)dZ_t + \xi(x_t, s_t, t)dJ_t$$

(3.25)

driven by a (continuous) Standard Brownian Motion $dZ_t$ and (discontinuous) Poisson or Compound Poisson jumps $dJ_t$. The spot price local drift $m(x_t, s_t, t)$, diffusion local volatility $v(x_t, s_t, t)$, and the predictable jump magnitudes $\xi(x_t, s_t, t)$ and jump probability intensities $\lambda(x_t, s_t, t)$ can depend on a vector of economic and environmental state variables $x_t$ (with their own Ito or jump-diffusion dynamics $dx_t$)

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16 The discussion here is exposited in terms of the physical availability of securities with which to trade the state variables and jumps. A market can also be epistemologically incomplete if sufficient securities are physically traded, but their dynamics are not known to the individual wanting an option pricing model. In dynamically complete markets, simply knowing the current prices of traded securities is not sufficient for a market to be dynamically complete. The future dynamics of the traded securities must also be known in order to recover the implicit underlying RN dynamics. This generalizes the point made regarding the RN identification in (3.24) in §3.3.
as well as on the spot price \( s_t \) and time \( t \).\(^{17}\) In such a setting, option prices are functions of \( s_t, x_t, \) and \( t \), along with whatever terms are required to reflect market preferences.

Options can still be valued in incomplete markets, but they cannot be priced via arbitrage. The problem with non-traded randomness in spot price dynamics is that there are no market prices from which to infer the market preferences which adjust the objective probabilities into RN probabilities. Equivalently, we cannot construct riskless hedged option+underlying portfolios as in the Black-Scholes and Black RN identifications. In a dynamically incomplete market, option pricing models therefore impose specific structure on market risk preferences in order to identify (compute) the RN dynamics. These so-called \textit{price-of-risk (POR)} assumptions specify the difference between objective and RN dynamics for the underlying commodity prices. The same POR is then assumed to be consistently embedded in all derivative prices. One model of incomplete market option pricing differs from another model because of the different specific POR assumptions they make. This issue is important for commodity option pricing. As will be seen in Chapter 4, a number of popular commodity option pricing models rely implicitly on POR assumptions.

Suppose again that the underlying commodity is a non-asset and that, while there may be observed market \textit{prices} for some commodity derivatives, there is no traded derivative whose price \textit{function} is known a priori. In this case, there is not enough information to pin down the RN spot price dynamics, and derivatives can only be priced given an auxiliary POR assumption. For example, if objective spot price dynamics are given by (3.20), then a common pricing procedure is to assume

\(^{17}\)Jump processes lead to discontinuities in the spot price process and possibly in the state variables. The \( t^- \) notation means that local drifts, volatilities and the jump moments do not depend on the spot price \( s_t \) and the state variables \( x_t \) at exactly date \( t \) (which would include any unpredictable jumps in the state variables occurring at date \( t \)), but rather on the limiting values \( \lim_{\tau \uparrow t} s_{\tau} \) and \( \lim_{\tau \uparrow t} x_{\tau} \) of the spot price and state variables as they approach date \( t \). The realized magnitudes of the jumps \( dJ_t \) can be either known in advance (e.g., normalized to 1 so that \( \xi(x_t, s_t, t) \) is the jump size) or they can be i.i.d. random variables (normalized with a mean 1 so that \( \xi(x_t, s_t, t) \) is the conditional expected jump size). See §4.1 in Chapter 4 and Shreve [197] Chapter 11 for more on Poisson and Compound Poisson processes.
Commodity Option Valuation

RN dynamics:

\[ ds_t = m^{RN}(s_t, t) \, dt + v(s_t, t) \, dZ_t \]  (3.26)

where the RN drift is

\[ m^{RN}(s_t, t) = m(s_t, t) + \pi(s_t, t) \]  (3.27)

where the price-of-risk \( \pi(s_t, t) \) is typically assumed to have a functional form that makes the RN dynamics qualitatively similar to the objective dynamics (which can be empirically estimated), but which must be quantitatively calibrated in some way (see §3.5). In contrast to RN asset drifts, non-asset RN drifts have no necessary relation to the risk-free interest rate \( r \).

Although markets cannot be dynamically complete with just a non-investible commodity and risk-free t-bills, once we can condition on an assumed POR, options can be priced and the market may be complete given tradable derivatives. Given a particular POR in (3.27), and the assumption of no jumps in (3.26), traded options with a non-zero sensitivity \( \partial c_t / \partial s \) can be used to dynamically synthesize and price non-asset commodity price randomness.\(^{18} \) Of course, this approach only works if the POR assumption is correct. Thus, dynamic replication based on an assumed ad hoc POR is subject to potential model error risk.

### 3.5 Calibration

Once the functional form of the RN dynamics has been specified, it still must be parametrically calibrated. There are two ways RN dynamics are calibrated. If parameters are the same under both the objective and RN measures, then one approach is to estimate them statistically using historical data. This, of course, assumes that the objective data generating process in the future is the same as it was in the past. Statistically estimated parameters are also estimated with error.

A widely-used alternative approach is implied calibration. If \( \Xi \) (pronounced “Xi”) denotes the parameters of the RN dynamics,

\[ ds_t = m^{RN}(s_t, t; \Xi) \, dt + v^{RN}(s_t, t; \Xi) \, dZ_t, \]  (3.28)

\(^{18}\)This result can be further generalized to allow for RN spot price dynamics that depend on multiple factors \( x_t \) driven by additional Brownian Motions and a set of options with non-zero linearly independent sensitivities \( \partial c_t / \partial s \) and \( \partial c_t / \partial x \).
then the prices of traded commodity-contingent securities depend implicitly on $\Xi$

$$c_t(\Xi) = \mathbb{E}_t^{RN} \left[ c_T^* \mid \Xi \right] e^{-r(T-t)}.$$  \quad (3.29)

Implied calibration involves solving numerically for values of $\Xi$ that equate one or more model prices $c_t(\Xi)$ with observed market security prices.

### 3.6 Summary

This chapter reviews how commodity RN price dynamics are identified and the connection with market completeness. A key idea implicit in this discussion is what defines a “state” for commodity price dynamics. The generalized Black-Scholes model assumes that the spot price and date pair $(s_t, t)$ constitutes the only state variables necessary for describing RN commodity spot price dynamics. Given this assumption, any $\mathcal{F}_T$-measurable commodity option payoff at any expiration date $T$ can be priced. In contrast, a coarser set of states in the single-futures Black model are defined based on time and the futures price for a particular fixed delivery date $\tau$. Given this coarser definition of the underlying state variables, $(F_t, \tau, t)$, the single-futures Black model can value option payoffs that are measurable with respect to a restricted filtration induced just by the particular futures price $F_{t, \tau}$ for a particular delivery date $\tau$.

When the spot price (or futures price) dynamics include non-asset randomness, then POR assumptions are usually needed for the identification of non-asset RN dynamics. This issue is discussed further in the context of specific multi-factor models in Chapter 4.

### 3.7 Notes

The existence, uniqueness, and identification of state prices, or equivalently RN probabilities, is more general than the application to commodities. In particular, the same identification issues arise when pricing options in incomplete markets for other non-asset underlying variables. This includes option valuation when the underlying variables are interest rates, weather, volatility, and events such as corporate defaults.
The general theory for dynamic state-contingent claim valuation was first worked out in Harrison and Kreps [116]. Duffie [83] is a more recent exposition. Continuous-time models of option pricing rely on the mathematics of stochastic calculus. Shreve [197] provides a systematic introduction to Brownian motions, Ito processes, jump processes, and their application to option pricing. When option valuations do not have analytic expressions, option prices are computed numerically either using tree/lattice and finite difference methods or via Monte Carlo simulation. Hull [123] describes basic tree/lattice and finite difference methods. Glasserman [105] is a comprehensive reference on Monte Carlo methods in finance.

Our discussion of option pricing in dynamically incomplete markets implicitly assumed that the POR for non-asset randomness only depends on commodity prices and time. In contrast, the PORs in equilibrium commodity option pricing models, as discussed in Chapter 4, may also depend on macroeconomic factors (e.g., aggregate wealth) which may also influence investor risk-bearing.

Our discussion of option pricing in incomplete markets also implicitly assumes that introduction of a new option (which needs to be priced) does not change the fundamental economics of valuation. This is a non-trivial assumption (see Allen and Gale [3]). In general, introducing new (non-redundant) securities in an incomplete market increases the span of tradable states which, in turn, can potentially change investor asset demands and, thus, can change market-clearing traded securities prices and their dynamics. In this case, the new option needs to be priced to be consistent with the new post-option-introduction traded security prices, not the pre-introduction prices.

A related set of issues concerns option pricing when liquidity is priced. In this case, the state prices implicit in actively traded securities may include a premium for liquidity which would not carry over to the valuation of illiquid real options. Vayanos and Wang [219] is a general survey on liquidity and asset pricing. Lastly, Black-Scholes and Black both assume trade occurs in frictionless markets (i.e., no bid-ask spreads and transaction costs). Option pricing with frictions is generally difficult. Kabanov and Safarian [127] discuss option pricing (and consumption-investment) models with transaction costs in trading.
Modeling Commodity and Energy Prices

This chapter reviews specific models used for commodity derivative valuation. Section 4.1 introduces the microeconomics of market-clearing commodity prices and their dynamics. Sections 4.2 and 4.3 review reduced-form spot price evolution models and reduced-form modeling of the term structure of futures prices. In practice, these are the most common approaches used to model commodity prices. Section 4.4 gives an overview of a third modeling approach, equilibrium pricing, which is less tractable but more conceptually grounded in the underlying economics of commodities. Section 4.5 gives a brief review of the empirical evidence on commodity prices. Section 4.6 summarizes. Section 4.7 gives general pointers to the literature.

4.1 Introduction

The valuation of options and derivatives of any type requires specifications for the dynamics of the underlying state variables driving the payoff cash flows. This is true both for financial options and commodity conversion assets. In the case of real options to physically store and transform commodities, the spot and futures prices at which commodi-
Commodity price processes are defined formally on a probability space $(\Omega, \mathcal{F}, P)$ where $\Omega$ is the set of all possible states, $\mathcal{F} = \{\mathcal{F}_t\}$ is a filtration which describes the possible evolution over time of information about the realized state $\omega \in \Omega$, and $P$ is a probability measure. More concretely, the dynamics of commodity prices are induced by the dynamics of supply and demand. In contrast to the more abstract treatment in Chapter 3, this chapter interprets specific models in terms of their ability to represent reasonable supply- and demand-driven commodity price dynamics.

From standard microeconomics, the spot price $s_t$ of a commodity at date $t$ is the market-clearing price which equates physical supply and demand. This is illustrated in the familiar diagram in Figure 4.1. The demand curve $D(s; x^D_t, t)$ is the aggregate quantity of the physical commodity demanded at each possible price $s$ on date $t$ given the set of factors $x^D_t$ which affect demand. Demand is the sum of demand for immediate physical use plus, if the commodity is storable, any inven-
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The supply schedule \( S(s; x^S_t, t) \) is the aggregate quantity of the physical commodity supplied at each possible price \( s \) on date \( t \) given the set of factors \( x^S_t \) which affect supply. Supply is the sum of current production plus any commodity stocks available from inventories stored in the past. The levels, slopes, and functional forms of the supply and demand curves can change over time depending on factors such as weather, macroeconomic conditions, and the prices of other commodities which are complements or substitutes in consumption and/or production. In equilibrium, aggregate supply equals aggregate demand at the equilibrium quantity, 

\[
D(s_t; x^D_t, t) = S(s_t; x^S_t, t) = Q(x_t, t),
\]

at the market-clearing price

\[
s_t = s(x_t, t), \tag{4.1}
\]

where \( x_t = (x^D_t, x^S_t) \) is the combined set of demand and supply factors and \( s(\cdot, \cdot) \) is the equilibrium spot price function. The dynamics of market-clearing spot prices \( ds_t \) are induced by the shapes of the supply and demand schedules, as reflected in the spot price function \( s(\cdot, \cdot) \), and by the dynamics of the supply and demand state variables \( dx_t \).

The stochastic processes followed by the state variables \( x_t = (x_1, \ldots, x_K, t) \) are typically represented as a system of stochastic differential equations like the following:

\[
dx_{k, t} = [m_k(x_{t-}, t) - \xi_k(x_{t-}, t) \lambda(x_{t-}, t)] dt + \nu_k(x_{t-}, t) dZ_t + \xi_k(x_{t-}, t) dJ_t, \quad \forall k = 1, \ldots, K, \tag{4.2}
\]

where the instantaneous change \( dx_{k, t} \) in the \( k \)-th state variable at date \( t \) is driven by potentially three building blocks: The deterministic passage of time \( dt \), the random instantaneous increments of a vector of Standard Brownian Motions \( dZ_t \), and the random instantaneous increments of a vector of Poisson or Compound Poisson jump processes \( dJ_t \) with jump probability intensities \( \lambda(x_{t-}, t) \). The corresponding jump magnitudes are either fixed (and normalized to 1 so that \( \xi_k(x_{t-}, t) \) is the vector of local jump sizes for the \( k \)-th state variable corresponding to the vector of jumps \( dJ_t \)) or are independently and identically distributed random vectors \( Y_t \) (with mean vector \( E_t[Y_t] \) normalized to the unit vector so that \( \xi_k(x_{t-}, t) \) gives the expected local jump sizes for the \( k \)-th state variable) defined at a discrete set of jump dates (i.e., dates on which
The passage of time $dt$ affects the state through changing seasons and other predictable changes (e.g., required lead times in bringing capacity on line or predictable components of macroeconomic or demographic variables). Since Brownian Motions are continuous, the increments $dZ_t$ represents the gradual arrival of incremental news. The jumps $dJ_t$ represent the arrival of dramatic news which causes discontinuous changes in the state variables and, thus, in prices.

The magnitudes of the impacts of the three building blocks $(dt, dZ_t, dJ_t)$ on each of the state variables $dx_{k,t}$ are determined by the expressions multiplying them: $m_k(x_{t-}, t)$ is the predictable local drift, $v_k(x_{t-}, t)$ is a vector of loadings which scale the volatility of the Standard Brownian Motions, and $\xi_k(x_{t-}, t)$ is a vector which scales the jump magnitudes. The local drifts, volatility, and jump intensities and magnitudes can all change over time depending on the state variables $x_{t-}$ and on time $t$. Time-dependence in price levels, volatilities, and other dynamics can be due to predictable seasonalities (e.g., harvests or weather patterns) and predictable macroeconomic, demographic, or technological trends (e.g., time to build new power plants or technology adoption dynamics). The $t^{-}$ notation means that drifts, volatilities and the jump moments do not depend on state variables at exactly date $t$ (which would include any unpredictable jumps in the state variables occurring at date $t$) but rather on the limiting value of the state variables as they approach date $t$. Including the $\xi_k(x_{t-}, t)\lambda(x_{t-}, t)$ term in the drift compensates for the contributions of the expected jumps so that $m_k(x_{t-}, t)$ is the total expected change.

The state variable dynamics $dx_t$ and the equilibrium spot price function $s(\cdot, \cdot)$ jointly induce the dynamics of the spot price $ds_t$. Figures 4.2 through 4.4 show historical daily spot prices for gold, natural gas, and power. The clear differences between the plots illustrate that the supply and demand dynamics for different commodities can differ sub-

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1 See Shreve [197] Chapters 3 and 11 for properties of Brownian motions and Poisson processes.
2 The total volatility of the state variables reflects both the scaled Brownian Motion randomness and the scaled jump randomness.
3 Including this adjustment as in (4.2) or excluding it as in (3.25) in Chapter 3 just affects the interpretation of the drift $m$. 
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Fig. 4.2 London Bullion gold spot prices between 2003 and 2012. Source: Federal Reserve Bank of St. Louis Economic Research (http://research.stlouisfed.org/fred2/series/GOLDPMGBD228NLBM; accessed Jan. 18, 2014).

stantially. Gold prices look like the prices of investible financial assets, such as equity, in that they appear to have persistent shocks that cause them to wander up or down. In contrast, natural gas prices appear to be mean-reverting, and power prices appear to be even more strongly mean-reverting. Other common empirical properties of spot prices for various commodities include seasonality in price levels, seasonality and other forms of heteroskedasticity in volatility, and jumps.

The statistical properties of commodity spot price processes have a microeconomic foundation in that they are induced by the statistical properties of the underlying supply and demand dynamics. This is illustrated in Figure 4.5. Factors which cause predictable seasonal changes in the levels of physical supply and demand (e.g., seasonal differences in average temperature or harvest cycles) can induce seasonalties in market-clearing spot price levels. Factors which cause supply and demand curves to fluctuate around a typical shape induce mean-reversion in spot prices (e.g., mean-reversion in temperature around a

Fig. 4.4 PJM Western Pennsylvania weighted average electricity spot price between 2003 and 2012. Source: US Energy Information Agency (http://www.eia.gov/electricity/wholesale/index.cfm; accessed Feb. 20, 2013).
seasonal average temperature or random equipment failures which initially reduce supply but then are repaired). Factors which cause supply and demand to become more or less elastic/inelastic (as shown in Figure 4.5) induce heteroskedasticity in spot prices. Factors which cause abrupt shifts in supply and demand (e.g., infrastructure failures) induce jumps in spot prices.

Different option pricing models make different assumptions about the number and type of state variables that drive the objective commodity price dynamics and the associated preference-adjusted RN dynamics. These different model assumptions lead to different option valuations. In particular, some models take the spot price $s_t$ and time $t$ as the sole state variables. Other models represent spot prices as a combination of multiple abstract statistical factors driving both price levels and price volatility. Still other models use explicitly identified factors – e.g., weather, equipment capacity, or business-cycle-driven demand – as causal state variables.

An important property of a commodity is whether the good is priced as an investible asset or as a non-asset. As discussed in Chapter 3, assets
have RN expected returns equal to the risk-free interest rate, but non-asset RN drifts have no necessary connection to the risk-free rate. The distinction between asset and non-asset commodities will be central in Section 4.2.

**Forward and futures prices.** Forward contracts are financial derivatives which allow the *long* side of a forward trade to buy a commodity from the *short* side of the trade at a forward price $f_{t,T}$ to be paid on a future delivery date $T$. In particular, the market forward price $f_{t,T}$ on date $t$ is contractually set so that both the long and short sides are willing to enter into the forward contract with no money changing hands at date $t$. There are no subsequent cash flows until delivery when the contract is settled with an exchange of value (from the long’s perspective) of $s_T - f_{t,T}$. This exchange of value can be through physical delivery of the commodity itself or via a cash settlement through a net cash payment.

Futures contracts, as discussed in §2.2 in Chapter 2, are similar to forward contracts except that futures settle on each day $t_i \leq T$ with a cash flow between the long and short sides equal to the difference $F_{t_i,T} - F_{t_i-1,T}$ between the prevailing futures price on day $t_i$ for delivery at $T$ and the same futures price from the day before. Thus, the only contractual difference between futures and forwards is the timing of settlement. A useful theoretical result in this context is that if the risk-free interest rate is non-random—an assumption which is often made for commodity option pricing given that commodity price volatility is empirically much larger than interest rate volatility—then arbitrage-free futures and forward prices are identical.4

On any given date $t$, the *futures curve (or futures term structure)* is the collection $E_t = \{F_{t,T}\}_T$ of futures prices for all traded future delivery dates $T$. If the futures curve is rising with time-to-delivery, then the futures term structure is said to be in *contango*. If the curve is falling in time-to-delivery, then the curve is said to be *backwardated*. In the natural gas market, it is common for futures curves to be both increasing and decreasing between various different delivery dates be-

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4 See Hull [123] Chapter 5 for more on futures and forwards.
cause of predictable seasonality in weather-driven peak demands for natural gas in winters (for heating) and sometimes in summers (for air conditioning).

Figure 4.6 shows the evolution of daily natural gas futures curves over the decade of 2003-2012. As time \( t \) passes and information flows into the market, the futures price \( F_{t,T} \) for a given delivery date \( T \) changes to reflect the impact of arriving persistent information. The seasonal humps in the natural gas futures curves create oncoming “waves” as time \( t \) passes and the time-to-delivery \( T - t \) gets shorter for fixed delivery dates \( T \). The impact of increased demand for natural gas over the decade and the impact of the shale gas boom at the end of the decade are readily apparent in the changing average price levels over time.

Futures prices can differ from expectations \( \mathbb{E}_t[s_T] \) of future spot prices, under the objective measure, because of risk premia. For ex-
ample, Keynes [138] describes how imbalances between heterogeneous speculators and hedgers can lead to premia and discounts in futures prices relative to objective future spot price expectations. In this case, the expected change in the futures price for a fixed delivery date $T$ can have a non-zero statistical drift. In contrast, under the RN measure, futures prices are RN expectations of future date $T$ spot prices conditional on date $t$ information,

$$F_{t,T} = E_t^{RN}[s_T],$$  \hspace{1cm} (4.3)

and, thus, futures prices are martingales under the RN measure:

$$E_t^{RN}[dF_{t,T}] = 0.$$  \hspace{1cm} (4.4)

The martingale property of RN futures price dynamics follows from iterated expectations and the daily settlement mechanics.

### 4.2 Spot-price Evolution Models

One widely-used type of reduced-form model is the spot price evolution approach in which the RN spot price dynamics are modeled without explicit reference to the underlying microeconomic drivers.

**Black-Scholes and further generalizations.** In §3.3 in Chapter 3, we saw that the local Black-Scholes drift can be any function $\mu(s_t, t)$ under the objective measure, but that the RN drift is the risk-free rate $r$ (see equations (3.7) and (3.15)). While possibly a reasonable approximation for stock prices and investible commodities (e.g., gold), these dynamics are problematic for non-asset commodities. In particular, the generalized Black-Scholes RN dynamics have constant expected returns equal to the risk-free interest rate, no seasonality, and no jumps. In contrast, actual commodity spot prices often exhibit mean-reversion, seasonalities, and jumps. While these statistical properties are characteristics of the objective dynamics observed in real-world price data, they also seem to carry over to the RN dynamics, since, in practice, it is often difficult to calibrate the generalized Black-Scholes model to reproduce seasonality in market futures curves and/or the term structures of volatility implicit in commodity option prices.
4.2. Spot-price Evolution Models

The Black-Scholes model has strong implications for the futures curve. Since the RN drift of spot prices is the risk-free rate, futures curves implied by Black-Scholes are always in contango and rise with time-to-delivery at the risk-free rate:

$$F_{t,T} = s_t e^{r(T-t)}.$$  \hspace{1cm} (4.5)

However, as can be seen in Figure 4.7, commodity futures prices in the market can increase and decrease with time-to-delivery in ways that are at odds with the Black-Scholes contango curves. The Black-Scholes model also restricts random shocks to the futures curves to proportional parallel shifts under the RN measure. To see this, use Ito’s Lemma and equations (4.5) and (3.15) to get the RN futures price dynamics

$$dF_{t,T}/F_{t,T} = v(s_t, t) dZ_t,$$  \hspace{1cm} (4.6)

where neither the local spot volatility $v(s_t, t)$ (from (3.7) in Chapter 3), which gives the local futures volatility for each maturity $T$, nor the

Fig. 4.7 NYMEX natural gas futures curve shapes on different trading dates. Prices are shown for the first twenty-four maturities. The three groups of backwardated, contango, and strongly seasonal daily futures curves are from the weeks starting on January 23, 2003, July 28, 2009, and June 25, 2007, respectively. Source: price-data.com (http://www.price-data.com/product/historical-futures-data/).
Standard Brownian Motion shock $dZ_t$ depend on the futures delivery date $T$ and, thus, both are the same for each futures price in the futures curve.

Subsequent research in commodity option pricing has worked to develop tractable models with more realistic commodity price dynamics. One variant of the Black-Scholes RN dynamics includes a deterministic local convenience yield, denoted by $y(t)$, in the spot price drift:

$$
ds_t = [r - y(t)] s_t \, dt + v(s_t, t) s_t \, dZ_t. \quad (4.7)$$

The convenience yield $y(t)$ is like a non-monetary pseudo-dividend representing a flow of services assumed to accrue to the owners of physical commodities. This variant of Black-Scholes can then be calibrated to fit any initial commodity futures curve by recursively choosing the convenience yields to set RN expected spot prices at each future delivery date equal to the corresponding market futures price:

$$
F_{t,T} = s_t e^{r(T-t)-\int_t^T y(\tau) \, d\tau}. \quad (4.8)
$$

If the convenience yield is greater than the risk-free rate, $y(T) > r$, then the futures curve will be locally decreasing in maturity at delivery date $T$ (i.e., the slope of the futures curve at delivery date $T$ will be negative), and if $y(T) < r$, then the futures curve will be locally increasing in maturity at delivery date $T$.\footnote{The futures curve will be backwardated over a range of delivery dates between $T_1$ and $T_2 > T_1$ if the integrated convenience yield $\int_{T_1}^{T_2} y(\tau) \, d\tau$ is greater than the corresponding deannualized risk-free rate $r(T_2 - T_1)$. Otherwise, it will be in contango.} However, conditional on the level of the realized spot price $s_t$, the associated futures curve on date $t$ is not random. In addition, random shocks to the spot price still cause random proportional parallel shifts in the futures curve.

The Black-Scholes model also has strong implications for the term structure of volatility. Consider two dates $T_1 > t$ and $T_2 > T_1$. Using the fact that Black-Scholes prices, given (4.7), are non-negative, the cumulative log spot price return variance over the time interval $T_2 - t$ can be decomposed as follows:

$$
\text{var}_t \left[ \ln \left( \frac{s_{T_2}}{s_t} \right) \right] = \text{var}_t \left[ \ln \left( \frac{s_{T_2}}{s_{T_1}} \right) + \ln \left( \frac{s_{T_1}}{s_t} \right) \right]. \quad (4.9)
$$
4.2. Spot-price Evolution Models

\[ \var_t \left[ \ln \left( \frac{s_{T_2}}{s_{T_1}} \right) \right] + 2 \text{cov}_t \left[ \ln \left( \frac{s_{T_2}}{s_{T_1}} \right), \ln \left( \frac{s_{T_1}}{s_t} \right) \right] + \var_t \left[ \ln \left( \frac{s_{T_1}}{s_t} \right) \right] \]

where

\[ \text{cov}_t \left[ \ln \left( \frac{s_{T_2}}{s_{T_1}} \right), \ln \left( \frac{s_{T_1}}{s_t} \right) \right] = \text{cov}_t \left[ \int_{\tau=T_1}^{T_2} d \ln(s_{\tau}), \ln \left( \frac{s_{T_1}}{s_t} \right) \right] \]

\[ = \text{cov}_t \left[ \int_{\tau=T_1}^{T_2} [r - y(t) - \frac{1}{2} v^2(\tau, s_{\tau})] d\tau + \int_{\tau=T_1}^{T_2} v(\tau, s_{\tau}) dZ_{\tau}, \ln \left( \frac{s_{T_1}}{s_t} \right) \right] \]

\[ = -\frac{1}{2} \text{cov}_t \left[ \int_{\tau=T_1}^{T_2} v^2(\tau, s_{\tau}) d\tau, \ln \left( \frac{s_{T_1}}{s_t} \right) \right]. \] (4.10)

The second equality in (4.10) follows from (4.7) and Ito’s Lemma, and the third equality follows because \( r - y(t) \) is non-random and because Brownian Motion increments \( dZ_{\tau} \) over \([T_1, T_2]\) are independent of information known at date \( T_1 \). Thus, the cumulative log return variance is increasing in time – that is to say, \( \var_t \left[ \ln \left( \frac{s_{T_2}}{s_{T_1}} \right) \right] > \var_t \left[ \ln \left( \frac{s_{T_1}}{s_t} \right) \right] \)

- if \( \text{cov}_t \left[ \int_{\tau=T_1}^{T_2} v^2(\tau, s_{\tau}) d\tau, \ln \left( \frac{s_{T_1}}{s_t} \right) \right] \) is zero (as with GBM or deterministic local volatility \( v(t) \)), negative, or not too strongly positive.

Monotone volatility can lead to difficulties in calibrating the Black-Scholes model to match objective volatility term structures and/or implied volatility term structures in option prices which can sometimes be non-monotone.\(^6\)

Another even more general variant of Black-Scholes allows the convenience yield to depend on the prevailing spot price level as well as on time:

\[ ds_t = [r - y(s_t, t)] s_t \, dt + v(s_t, t) s_t \, dZ_t. \] (4.11)

Now the shape of the futures curve can change stochastically in ways that are empirically more realistic. For example, high spot prices can be associated with local backwardation at the short end of the futures curve.

\(^6\)If the objective and RN commodity prices are instead modeled as mean-reverting processes (negatively autocorrelated), then heteroskedasticity and/or changing speeds of mean-reversion can lead to non-monotone cumulative log return variance in the model. In particular, non-monotonicity is possible even if the local volatility function is constant or deterministic.
curve if \( y(s_t, t) > r \) when \( s_t \) is high and with local contango if \( y(s_t, t) < r \) when \( s_t \) is low. However, this randomness is driven entirely by the spot price, which means, again, that knowing \( s_t \) and the date \( t \) is sufficient to know the shape of the futures curve.

While convenience yields improve the fit of Black-Scholes to market commodity derivative prices, there are two conceptual problems with this modeling approach. First, when the underlying variable is an investible asset, the Black-Scholes RN dynamics are derived from a no-arbitrage argument without POR assumptions (see §3.3 in Chapter 3). However, the no-arbitrage argument is valid only if the underlying variable is an investible asset. Consequently, when the various variants of Black-Scholes are applied to non-asset commodities, the RN dynamics in (3.15), (4.7), and (4.11) are effectively assumed, not derived. In this sense, while the mathematics of the RN dynamics can be assumed to have a generalized Black-Scholes form, the conceptual justification for these RN dynamics is absent when the Black-Scholes model is used to price non-asset commodity options.

The second problem specifically concerns convenience yields. Unlike stocks which pay actual dividends, ownership of a barrel of oil or an mmBtu of natural gas does not pay any interim cash flows to commodity owners. Commodity ownership may entail storage costs (negative dividends), but no positive cash flows accrue to owners of commodities over time. Similarly, a short position in equity involves paying dividends, but short positions in commodities do not receive any dividends when convenience yields are negative. In short, convenience yields may be a convenient model-fitting device, but pseudo-dividends have no physical basis in cash flows in the real world to put any restrictions on the commodity RN drift.

The rest of this section presents a variety of RN spot price dynamics used in other option pricing models which have been proposed as alternatives to Black-Scholes given its potential problems as a model of non-asset commodity price dynamics.

**Gibson-Schwartz.** The Gibson and Schwartz [104] model is a well-known two-factor model of RN spot price dynamics:

\[
\begin{align*}
    ds_t &= [r - y_t] s_t \, dt + \sigma s_t \, dZ_{s,t},
\end{align*}
\] (4.12)
4.2. Spot-price Evolution Models

\[ dy_t = \alpha[\theta - y_t] dt + \xi dZ_{y,t}. \]

Spot prices follow a process similar to a Geometric Brownian Motion except that now a stochastic convenience yield is subtracted from the the risk-free rate in the RN spot price drift. The convenience yield follows an Ornstein-Uhlenbeck process which mean-reverts to a long-run RN mean of \( \theta \) which includes both the objective long-run mean and a possible POR adjustment. The objective and RN speeds of mean-reversion can also differ depending on the POR assumption. The Standard Brownian Motion \( dZ_{s,t} \) represents persistent shocks to commodity supply and demand, whereas \( dZ_{y,t} \) changes how much the mean-reverting convenience yield causes spot prices to deviate from constant-drift paths. The two shocks, \( dZ_{s,t} \) and \( dZ_{y,t} \), can also be correlated.

In terms of fitting observed market derivative prices, the Gibson-Schwartz model has additional flexibility relative to the deterministic and the Markov convenience yield versions of Black-Scholes in (4.7) and (4.11). Given the spot price \( s_t \), futures prices in Gibson-Schwartz also depend on the realized convenience yield \( y_t \) which is now random. Thus, futures prices are not perfectly correlated with prevailing spot price, and the shape of the futures curve at date \( t \) is random given just the spot price at date \( t \). The short end of the futures curve will be locally increasing (decreasing) if the current convenience yield \( y_t \) is less (greater) than the risk-free interest rate \( r \). Whether or not the futures curve is increasing or decreasing at distant future delivery dates depends on whether the long-run mean convenience yield \( \theta \) is less than or greater than \( r \). Once again, as in the generalized Black-Scholes model, since the RN commodity price process appears to have a RN drift equal to the risk-free rate after adjusting for the convenience yield – i.e., the same RN drift as for asset prices – the implicit suggestion that commodity prices are asset prices is a misleading intuition for non-asset commodities.

**Affine models.** Schwartz [186] extends the two-factor Gibson and Schwartz [104] model to a three-factor model which also includes a spot interest rate process. Casassus and Collin-Dufresne [48] further extend the specification of multi-factor spot rate models and investigate their empirical fit. In particular, they assume that log commodity spot
prices under the RN measure are generated by a maximal affine system with three state variables.\footnote{Dai and Singleton \cite{64} define an affine model as being “maximal” if it has the maximum number of identifiable parameters given futures prices alone.} Stochastic risk-free spot interest rates are assumed to follow a mean-reverting Ornstein-Uhlenbeck process:
\begin{equation}
    dr_t = \alpha_r [\theta_r - r_t] dt + \sigma_r dZ_{r,t},
\end{equation}
and the underlying commodity is implicitly modeled as an investible asset with RN dynamics for the spot price and convenience yield:
\begin{align}
    ds_t &= [r_t - y_t] s_t dt + \sigma_s s_t dZ_{s,t}, \quad (4.14) \\
    dy_t &= [\kappa_{y,0} + \kappa_{y,r} r_t + \kappa_{y,y} y_t + \kappa_{y,s} \ln(s_t)] dt + \sigma_y dZ_{y,t},
\end{align}
where $\alpha_r, \theta_r$ are model parameters. In this model the convenience yield is not just correlated with the spot price (i.e., via potentially correlated Standard Brownian Motions $dZ_{y,t}$ and $dZ_{s,t}$); rather now the convenience yield dynamics depend directly on spot prices as a factor in the convenience yield drift.

\textbf{Pilipovic/Schwartz-Smith.} Pilipovic \cite{171} and Schwartz and Smith \cite{187} dispense with treating imperfectly storable commodities as investible assets with RN expected returns equal to the risk-free interest rate.\footnote{Ross \cite{180} models RN spot prices with mean-reversion but no persistent shocks.} Instead, they simply posit an incomplete market setting in which the RN spot price dynamics include permanent and transitory components:
\begin{align}
    s_t &= L_t \epsilon_t, \quad (4.15) \\
    dL_t &= \mu^{RN} L_t dt + \sigma L_t dZ_{L,t}, \\
    d\epsilon_t &= \alpha_t \theta_t dt + \xi dZ_{\epsilon,t},
\end{align}
where the two Standard Brownian Motions, $dZ_{L,t}$ and $dZ_{\epsilon,t}$, are potentially correlated. The Geometric Brownian Motion $L_t$ represents persistent long-run price trends due to persistent shocks to long-term supply and demand (e.g., technological innovations or discoveries of new deposits). The mean-reverting process $\epsilon_t$ then causes the market-clearing spot price to wander away from the long-term price due to short-lived
supply and demand shocks (e.g., weather). The RN dynamics in (4.15) may differ from the objective dynamics depending on implicit POR assumptions (as discussed in Section §3.4 in Chapter 3).

Futures prices are available in closed-form:

\[ F_{t,T} = L_t \exp \left\{ e^{-\alpha(T-t)} \epsilon_t + [1 - e^{-\alpha(T-t)}] \theta + A(T-t) \right\} \]  

(4.16)

where \( A(T-t) \) is a known function of the futures time-to-maturity that depends on the other parameters of the model. Note that, although \( L_t \) and \( \epsilon_t \) are unobserved latent variables, once the model parameters are known, the two factors can be recovered (i.e., implied) from any two futures prices.

One significant fact – proven in Schwartz and Smith [187] – is that the Gibson-Schwartz and Pilipovic/Schwartz-Smith models are mathematically equivalent “reformulations” of each other. For each calibration of the Pilipovic/Schwartz-Smith RN dynamics in (4.15), there is a mathematically equivalent calibration of the Gibson-Schwartz dynamics in (4.12) and vice versa. Thus, the critique of the dividend interpretation of the Gibson-Schwartz convenience yield is not a criticism of the mathematics per se, but rather a critique of the economic motivation for the mathematics and its connection to real-world economics. In contrast, the Pilipovic/Schwartz-Smith model does not suggest that commodity prices have anything to do with asset prices. Rather, it simply starts with the premise that RN commodity price dynamics have persistent and transitory components similar to those often seen in actual prices for many commodities.

This model is easily adapted to include multiple transitory factors; e.g., two factors as in:

\[
\begin{align*}
    s_t &= L_t e^{\epsilon_{1t} + \epsilon_{2t}}, \\
    dL_t &= \mu^{RN} L_t \, dt + \sigma L_t \, dZ_{L,t}, \\
    d\epsilon_{it} &= \alpha_i [\theta_i - \epsilon_{it}] \, dt + \xi_i \, dZ_{i,t}, \quad i = 1, 2
\end{align*}
\]  

(4.17)

where the speeds of mean reversion operate on different time scales, \( \alpha_1 \neq \alpha_2 \). For example, \( \epsilon_{1t} \) may be a slowly mean-reverting process due to business cycle fluctuations, while \( \epsilon_{2t} \) could be a faster mean-reverting process due to weather.
One limitation of the Pilipovic/Schwartz-Smith model is that it is not automatically consistent with an arbitrary initial market futures curve. To the extent that it implies that current market futures prices are inconsistent with each other given the RN model dynamics, we say that the model is not \textit{arbitrage-free}.\footnote{Since Gibson-Schwartz is mathematically equivalent to Pilipovic/Schwartz-Smith, the same issue applies there too.} However, the model can be made arbitrage-free if it is modified to include a deterministic time-dependent drift. For example, if the mean-reverting drift pulls the transitory component towards different levels $\theta(t)$ over time as in

\begin{align*}
  s_t &= L_t e^{\epsilon_t}, \\
  dL_t &= \mu^{RN} L_t \, dt + \sigma L_t \, dZ_{L,t}, \\
  d\epsilon_t &= \alpha [\theta(t) - \epsilon_t] \, dt + \xi \, dZ_{\epsilon,t},
\end{align*}

then $\theta(t)$ can be calibrated to ensure that the RN expected future spot prices agree with any current market futures curve.

\textbf{Jaillet-Ronn-Tompaidis.} The Jaillet-Ronn-Tompaidis [125] is a one-factor mean-reverting RN spot price model with a deterministic seasonal price level:

\begin{align*}
  s_t &= g(t) e^{\epsilon_t}, \\
  d\epsilon_t &= \alpha [\theta(t) - \epsilon_t] \, dt + \sigma dZ_t.
\end{align*}

In this specification there are no persistent random shocks to prices. All of the spot price randomness comes from short-term temporary deviations away from seasonally-adjusted normal prices given the seasonal scaling factor $g(t)$. The associated futures prices are given by

\begin{equation}
  F_{t,T} = g(t) \exp \left\{ e^{-\alpha(T-t)} \epsilon_t + \left[ 1 - e^{-\alpha(T-t)} \right] \theta + \left[ 1 - e^{-2\alpha(T-t)} \right] \frac{\sigma^2}{4\alpha} \right\}.
\end{equation}

The time-dependent scaling function $g(t)$ allows the model to be calibrated to match the market futures curve. Once again, the RN and objective dynamics may differ depending on specific POR assumptions.

\textbf{Stochastic volatility.} An alternative to price- and time-dependent heteroskedasticity are spot price models with a stochastic volatility
factor as in\(^{10}\)
\[
\begin{align*}
    ds_t &= \alpha[\theta_s(t) - s_t] \, dt + v(x_t) \, s_t \, dZ_{s,t}, \\
    dx_t &= \kappa[\theta_x(t) - x_t] \, dt + \sigma dZ_{x,t}.
\end{align*}
\] (4.21)

The spot prices here are mean-reverting with seasonal mean price levels and the local volatility is a function of a random volatility factor \(x_t\). The volatility factor could reflect randomly changing slopes of the supply and demand curves or randomly changing underlying weather or economic factor volatilities. The volatility factor mean-reverts to a time-dependent level \(\theta_x(t)\) at speed \(\kappa\) and has a volatility of volatility (“vol vol”) of \(\sigma\). Although the volatility factor follows an Ornstein-Uhlenbeck process, which can go negative, the function \(v(\cdot)\) can be specified to keep spot price volatility non-negative. The time-dependent price level \(\theta_s(t)\) lets the process match any futures price term structure, and the seasonal volatility level \(\theta_x(t)\) lets the process match any volatility term structure.

**Mean-reverting jump diffusion (MRJD).** Jumps are an important feature of daily (and intra-day) power price dynamics. For example, Figure 4.8 shows how jumps in electricity prices can occur. Power plant or transmission failures cause the aggregate short-term supply curve to shift back to the left (from the solid supply curve to the dashed curve) since less supply is available, which can cause the intersection with the short-run demand curve to jump up discontinuously. The more inelastic (steeper) the demand curve is, the larger the price jumps are.

Over time, as the grid operator finds new sources of power, as repair crews fix broken equipment, and as price-sensitive users curtail their power use, the supply and demand curves adjust causing prices eventually to mean-revert back towards the old seasonal equilibrium price and quantity.

The MRJD diffusion model captures these various qualitative features:
\[
\begin{align*}
    ds_t &= \alpha[\theta(t) - s_{t-}] \, dt + v(s_{t-}, t) \, dZ_t + \xi(s_{t-}, t) \, dJ_t.
\end{align*}
\] (4.22)

\(^{10}\)Our discussion of stochastic volatility models is generic. Specific models include Heston [119] and SABR (see Hagan, Kumar, Lesniewski, and Woodward [114]).
Prices wander around a seasonal price level $\theta(t)$ driven by heteroskedastic Brownian Motion shocks and Poisson jumps with state-dependent jump intensities $\lambda(s_{t-}, t)$ and jump magnitudes $\xi(s_{t-}, t)$. When the RN commodity price dynamics are assumed to follow a MRJD process, this usually involves implicit POR assumptions.

### 4.3 Futures Term Structure Models

Another widely-used approach for commodity option valuation directly models the RN dynamics of the entire term structure of futures prices. There are two general points to make here. The first is that the dynamics of the futures curve represent the process of arriving new information and the intertemporal correlations between supply and demand at different delivery dates. Different types of information can have different impacts on spot price expectations for different future delivery dates. Some events cause persistent shifts to future supply and demand over time which cause the whole term structure of future prices to shift up or down (e.g., development of new technologies like hydraulic fracking or increased construction of natural gas power plants). Other events have supply and demand effects which are more transitory and, thus, move the spot price and short-dated futures prices more than longer-dated futures prices (e.g., short-run weather patterns).
The second general point is that futures curve dynamics imply spot price dynamics and vice versa. For example, if futures prices for early delivery dates shift more than long-dated futures prices, that implies that the impact of that particular piece of news is transitory and that its effect on spot prices will predictably mean-revert away. Conversely, if a shock to spot prices is expected to mean-revert away over time, then that predictable decay should be reflected in longer-dated futures prices reacting less than short-dated futures prices. Although the two approaches are isomorphic, the term structure modeling approach has some modeling tractability advantages over the spot price modeling approach. In practice, the drift of the RN spot price process must typically include a calibrated time-dependent deterministic function – for example, $\theta(t)$ in the modified Pilipovic/Schwartz-Smith model (4.18) – to be arbitrage-free given the initial futures curve observed in the market. In contrast, since futures term structure models are specified directly in terms of changes relative to an initial futures curve, setting the initial curve equal to the observed current market futures curve automatically ensures that the future RN futures dynamics are consistent with the current futures curve. Thus, in contrast to some of the parsimonious spot price evolution models in §4.2, the HJM model is, by definition, arbitrage-free given the current market futures curve.

The modeling of futures curve dynamics draws heavily on modeling techniques first developed in an interest rate context. Heath, Jarrow, and Morton (HJM) [118] derived an arbitrage-free model of the RN dynamics for a continuum of instantaneous forward rates. Their key insight is that the RN drifts of forward rates are pinned down by the forward rate volatility function and the fact that forward rate dynamics must be consistent with RN expected bond returns being equal to spot interest rates. Brace, Gatarek, and Musiela [32] and Jamshidian [126] derive a version of HJM – called the market model – which is specified in terms of discrete-term forward rates rather than instantaneous forward rates. This modification was motivated by the fact that, in practice, fixed income traders work with finite sets of forward rates with discrete maturities. Kennedy [136, 135], Goldstein [109], and Longstaff, Santa-Clara, and Schwartz [153] develop and work with string models which have the same dimensionality as the set of forward rates. This lets the
forward rate model be consistent both with the initial forward rate curve and with the empirical correlation structure of forward rates.

There is, however, at least one significant difference between modeling term structures of forward rates and modeling term structures of commodity futures prices. Futures prices are RN martingales. Thus, some of the mathematical complications associated with specifying RN forward interest rate drifts that are consistent with RN bond price dynamics do not arise in a futures price term structure model. This can simplify commodity futures price term structure modeling.

**Black model.** The single-futures Black [20] model, discussed in §3.3 in Chapter 3, represents the RN dynamics of futures prices over time for a particular single fixed future delivery date $T$ as a driftless Ito process:

$$dF_{t,T} = v(F_{t,T}, t) F_{t,T} dZ_t.$$  

(4.23)

This can be extended to a model of futures curve dynamics if this same equation is assumed to hold for the RN dynamics for futures prices for all delivery dates $T$. In particular, if the Standard Brownian Motion increment is not specific to a particular maturity $T$ but rather affects all futures maturities, then the Black term structure model implies that changes in futures prices for all delivery dates are locally perfectly correlated due to their common dependence on $dZ_t$. In the special case of delivery-date-independent shocks and time- and spot-price-contingent local futures price local volatility,

$$dF_{t,T} = v(s_t, t) F_{t,T} dZ_t, \forall T,$$  

(4.24)

the futures curve has proportional parallel shifts. Comparing (4.24) with (4.6) shows that this version of an extended Black futures term structure model is consistent with the generalized Black-Scholes spot price model.

**Multi-factor term structure model.** The RN futures dynamics can be made richer than perfectly correlated local shifts by introducing multiple (i.e., $N$) factors and allowing futures prices for different delivery
4.3. Futures Term Structure Models

where the Standard Brownian Motions $dZ_{j,t}$ are cross-sectionally independent for all of the $N$ factors. The factor loading notation $v_j(t, x_t; T)$ is interpreted as follows: The dependence on $T$ in $v_j(t, x_t; T)$ means that futures prices for different delivery dates $T$ along the futures curve on a given date $t$ can respond differently to a particular shock $dZ_{j,t}$. The dependence on $j$ means that a given futures price can load differently on different shocks. The dependence on $t$ means that the sensitivity of a futures price for a given (fixed) delivery date $T$ to shock $dZ_{j,t}$ can change over time $t$ because of seasonalities or because factor sensitivities change as the time-to-delivery $T - t$ gets shorter. For example, the *Samuelson effect* – an empirical regularity that futures volatility tends to be greater for short time-to-delivery futures prices than for long time-to-delivery futures prices – is consistent with factor loadings which are decreasing in $T - t$. The dependence on $x_t$ allows factor sensitivities to depend, not just on deterministic time effects, but also on a random factor. If $x_t$ is a non-futures state variable, then the RN dynamics for $dx_t$ must also be specified. As needed, the factor loadings can be further generalized to allow for dependence on the futures term structure itself and/or on a set of multiple volatility factors.

Cortazar and Schwartz [62] is an example of this approach. In their model, the cross-delivery-date covariances depend on both $t$ and the delivery dates $T$. In the special case in which the factor loadings are stable over time and just depend on time-to-maturity $T - t$, the factor loadings can be empirically calibrated using Principal Component Analysis. We can further generalize this empirical approach to estimate seasonal factor sensitivities (i.e., where the factor sensitivities depend directly on $t$) by estimating separate PCAs for each season.

In practice, a small number of factors is often sufficient for a factor

\[ dF_{t,T} = \left[ \sum_{j=1}^{N} v_j(t, x_t; T) dZ_{j,t} \right] F_{t,T} \tag{4.25} \]

\footnote{The single-factor exponential term structure specification $dF_{t,T}/F_{t,T} = \sigma \exp[-\alpha(T - t)]dZ_t$ with $\alpha > 0$ in Clewlow and Strickland [60] is an example of such a decaying factor loading.}
model to reproduce a large part of the empirical variability of futures price term structures. Litterman and Scheinkman [151] show that level, slope, and curvature factors explain over 90 percent of interest rate term structure variability. Seccomandi, Lai, Margot, Scheller-Wolf, and Seppi [194] find that slope and curvature explain well over 90 percent of natural gas futures price term structure variability. Working with low-dimensional factor models can sometimes improve computational tractability for determining optimal operating policies for commodity conversion assets.

The dynamics of futures prices induce the dynamics of spot prices. To see this, consider the (natural) log spot price at any date \( t \) after the current date \( t_0 \):

\[
\ln s_t = \ln F_{t,t} = \ln F_{t_0,t} + \int_{\tau=t_0}^{t} d\ln F_{\tau,t}.
\]

From (4.25) and Ito’s Lemma, holding \( t \) fixed and allowing the time index \( \tau \) to vary over time between \( t_0 \) and \( t \) gives the RN dynamics for the futures price \( F_{\tau,t} \) with a fixed delivery date \( t \):

\[
d\ln F_{\tau,t} = -\frac{1}{2} \sum_{j=1}^{N} \int_{\tau=t_0}^{t} v_j^2(\tau, x_\tau; t) \, d\tau + \sum_{j=1}^{N} \int_{\tau=t_0}^{t} v_j(\tau, x_\tau; t) \, dZ_{j,\tau}.
\]

Substituting (4.27) into (4.26) gives

\[
\ln s_t = \ln F_{t_0,t} - \frac{1}{2} \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} v_j^2(\tau, x_\tau; t) \, d\tau \right] + \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} v_j(\tau, x_\tau; t) \, dZ_{j,\tau} \right].
\]

\[\text{See also Cortazar and Schwartz [62], Clewlow and Strickland [60, Chapter 8], Blanco et al. [22], Tolmasky and Hindanov [214], Geman and Nguyen [102], Borovkova and Geman [29], and Frestad [98] for other empirical factor model estimations with three factors. However, Manoliu and Tompaidis [157], Borovkova and Geman [28], Suenaga et al. [204], and Wu et al. [231] use fewer than three factors and Eydeland and Wolyniec [86, pp. 351-367], Gray and Khandelwal [111, 112], Bjerkund et al. [19], and Thompson [209, 210] use more than three factors.}
Applying Ito’s lemma again, now allowing the time index \( t \) to change, gives a stochastic differential equation for log spot prices:

\[
\begin{align*}
    d\ln s_t &= \left\{ \frac{\partial \ln F_{t_0,t}}{\partial t} - \frac{1}{2} \sum_{j=1}^{N} v_j^2(t, x_t; t) \right. \\
    &\quad \left. - \frac{1}{2} \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} \frac{\partial v_j^2(\tau, x_\tau; t)}{\partial t} \, d\tau \right] \\
    &\quad + \sum_{j=1}^{N} \left[ \int_{\tau=t_0}^{t} \frac{\partial v_j(\tau, x_\tau; t)}{\partial t} \, dZ_{j,\tau} \right] \right\} dt \\
    &\quad + \sum_{j=1}^{N} v_j(t, x_t; t) \, dZ_{j,t}
\end{align*}
\]

provided that all of the the factor loadings \( v_j(\tau, x_\tau; t) \) and their derivatives in \( t \) are continuous. The four components in the drift in (4.29) can be interpreted as follows: First, the slope of the past futures curve observed on date \( t_0 \) for the delivery date \( t \) reflects predictable (e.g., seasonal) changes in the spot price at date \( t \). The next two terms in the spot price drift are Ito adjustments (from Ito’s Lemma) due to the log transformation. The fourth term in the drift reflects any predictable autocorrelation in how spot prices respond to factor shocks over time. At date \( t \) the spot price \( s_t \) reflects the impact of factor shocks on prior dates \( \tau \) between dates \( t_0 \) and \( t \) as reflected in changes in prior futures prices \( F_{\tau,t} \) with delivery at time \( t \). Similarly, on date \( t + \Delta t \) (for a small \( \Delta t > 0 \)) the spot price \( s_{t+\Delta t} \) also reflects those same past factor shocks on dates \( \tau \) but as reflected in the changes in the prior futures prices \( F_{\tau,t+\Delta t} \) with delivery at time \( t + \Delta t \). The derivative \( \partial v_j(\tau, x_\tau; t)/\partial t \) represents the differential impact of those past shocks on the date \( t + \Delta t \) delivery futures prices relative to the impact on the date \( t \) delivery futures prices in the limit, i.e., as \( \Delta t \to 0 \). Lastly, to the extent that the integrals in the third and fourth terms in the drift depend on the

---

\[\text{Spot price changes } d\ln s_t \text{ can depend on current and past values of the factor } x_t, \text{ but they do not include contemporaneous } dx_t \text{ shocks. This is analogous to the stochastic volatility model (see (4.21)) in which the spot price change } ds_t \text{ does not include the volatility shock } dx_1.\]
realized sequence of factor values $x_t$ and Standard Brownian Motion shocks $dZ_{j,t}$ over time, the spot price process will be non-Markovian.\footnote{See Carverhill [47] who gives sufficient conditions for Markovian dynamics.}

**String models.** If the number of factors is smaller than the dimension of the variance-covariance matrix of the futures curve, then the factor model imposes restrictions on the futures curve variance-covariance matrix since the rank of this matrix is less than the number of futures prices in the futures curve. In practice, option traders sometimes want models which allow for some amount of maturity-specific variation at each point along the futures curve. The string model approach – drawing on Kennedy [136, 135], Goldstein [109], and Longstaff, Santa-Clara, and Schwartz [153] – allows for maturity-specific variability along the entire term structure:

\[
dF_{t,T} = v(t, x_t; T) F_{t,T} dZ_{t,T}, \forall T. \tag{4.30}
\]

This model looks similar to the proportional parallel shift Black term structure model (4.24) except that here there is a separate Standard Brownian Motion, $dZ_{t,T}$, corresponding to each maturity $T$ along the futures curve as opposed to a single Standard Brownian Motion $dZ_t$ driving the entire curve. However, rather than independent Brownian Motions driving different futures prices along the term structure, the different Brownian Motions in the string model are potentially correlated. One empirical advantage of a string model is that the correlation structure of factors in a string model for $N$ futures prices and constant correlations is exactly identified by the empirical correlation matrix. In particular, there are $(N-1)N/2$ correlations to be specified in the string model and the same number of distinct correlations in the empirical correlation matrix. In contrast, a factor model with $N$ factors has $N^2$ factor loadings. Consequently, additional structure must be assumed to identify an $N$ factor model.

### 4.4 Equilibrium Models

Spot price evolution models and futures price term structure models represent the reduced-form statistical properties of commodity price
dynamics. The focus in much of “reduced-form” modeling is to develop flexible forms which can be used to price real and financial commodity derivatives. Along side this work, a large body of research explicitly models the underlying economic drivers of commodity supply and demand and their impact on the equilibrium properties of commodity prices.

The early seminal work on equilibrium commodity pricing was by Kaldor [128], Working [227, 228], Telser [208], and Samuelson [183]. Later developments on commodity pricing are in Wright and Williams [229], Williams and Wright [226], Brennan [35], Deaton and Laroque [70, 71], Litzenberger and Rabinowitz [152], Routledge, Spatt, and Seppi [182], and Tayur and Yang [206] among others. Much of this work goes under the name theory of storage because of its focus on the dynamics of storage and how storage moderates the price impact of transitory – random or seasonal – shocks to commodity production and consumption.

The key idea in the theory of storage is that physical storage gives speculators an American-style timing option of when to use the commodity. The result is that prices and aggregate storage inventories are jointly determined. Buying by speculators at date $t$ raises spot prices at date $t$ (since less is available for current consumption) and lowers expected spot prices at later dates $t' > t$ (as future inventory withdrawals augment future production). The price impact of inventory affects the profitability of inventory at date $t$. In a competitive equilibrium, inventory at date $t$ adjusts the spot price at date $t$ and the probability distribution of future spot prices until either (i) speculators are indifferent about acquiring the final marginal unit of inventory or (ii) physical constraints on aggregate inventory levels or inventory changes (of the type considered in Chapter 5) bind. In particular, one important constraint is that inventory cannot be negative. When aggregate inventory hits zero, this is called a stock-out. In a stock-out, the futures curve will be backwardated. Even though investors might like to withdraw more of the commodity from inventory and sell it in the spot market (i.e., if current spot prices are above expected future spot prices), they are unable to do so once inventory is exhausted. In addition, the futures curve at later delivery dates can also be backwardated if the probability
of future stock-outs and, hence, future backwardation is high enough (since the current futures curve is the RN expectation of future futures curves at later dates). Thus, these equilibrium models provide a theory for backwardation. In addition, the theory of storage implies that commodity price volatility will tend to be high in stock-outs (when markets are backwardated) due to the absence or shortage of inventory to buffer demand and supply shocks.

Recent research has also modeled commodity production. Carlson, Khokher, and Titman [44] extend the theory of storage approach to the pricing and optimal extraction of an exhaustible commodity in which natural commodity deposits in the ground are viewed as a type of storage subject to extraction frictions. Casassus, Collin-Dufresne, and Routledge [49] show how lumpy irreversible investment in commodity production capacity (e.g., development of oil reserves) leads to regime-switching between different endogenous commodity price dynamics. Kogan, Livdan, and Yaron [141] show how irreversible investment in commodity production can cause non-monotonicities in the relation between the futures curve slope and volatility.

Equilibrium models of commodity prices often assume that the marginal consumers and investors in the market have risk-neutral preferences with respect to commodity prices. This may be a plausible assumption for commodities which are a relatively small part of the aggregate consumption bundle. However, it is possible that oil and other major commodities may themselves be systematic macroeconomic factors which directly affect pricing in financial markets and, thus, may have their own risk premia. In addition, the prices of even economically small commodities can still be affected by supply and demand correlations with other systematic macroeconomic factors. Recent research seeks to understand the role of preferences in commodity pricing and risk premia in commodity-linked investments. For example, motivated by dramatic changes in commodity prices in the mid 2000s, Baker and Routledge [7] show how changes in endowments and preferences can affect both the shape of the futures curve and the futures open interest in an endowment economy with heterogeneous agents, multiple con-

4.5 Empirical Research on Commodity Prices

There is a large literature on empirical commodity pricing. A thorough survey of all of this research is beyond the scope of this chapter, but we do give pointers to work on several big research questions. Our focus here is on general empirical issues and on tests of specific pricing models.

One major research question concerns whether there are risk premia in commodity futures prices. A common empirical methodology to investigate this question uses predictive regressions:

\[ s_{t+n} - s_t = a_0 + a_1 [F_{t,t+n} - s_t] + \epsilon_{t+n} \] (4.31)

where the change in the spot price between date \( t \) and date \( t + n \) (i.e., over \( n \) periods) is regressed on the date \( t \) futures-spot price basis, \( F_{t,t+n} - s_t \), for the futures price with delivery on date \( t + n \). The explanatory variable in the regression can be decomposed as follows:

\[ F_{t,t+n} - s_t = (F_{t,t+n} - \mathbb{E}_t^M [s_{t+n}]) + (\mathbb{E}_t^M [s_{t+n}] - s_t) \] (4.32)

where \( F_{t,t+n} - \mathbb{E}_t^M [s_{t+n}] \) represents any market risk premium (or discount) in future prices set at date \( t \) relative to the market’s expected future spot price and where \( \mathbb{E}_t^M [s_{t+n}] - s_t \) is the market’s belief about the objective expected change in the spot price based on date \( t \) information. If the risk premium is constant and if the market is informationally efficient – i.e., in that the market’s expectation \( \mathbb{E}_t^M [s_{t+n}] \)

---

15 Recursive preferences are a departure from expected utility preferences in that the utility \( u(C_t, V_t) \) at each date \( t \) depends on both current consumption \( C_t \) and on the value \( V_t \) of expected future utility. See Epstein and Zin [85], Kreps and Porteus [144], and the survey paper by Backus, Routledge, and Zin [6].

16 The \( \mathbb{E}_t^M \) notation allows for the possibility of market inefficiency if the market’s expectations differ from the true objective expectations. Since \( \mathbb{E}_t^M \) refers to the market’s beliefs about objective dynamics, it should not be confused with RN expectations \( \mathbb{E}_t^{RN} \). In fact, since futures prices are RN expectations, the risk-premium component \( F_{t,t+n} - \mathbb{E}_t^M [s_{t+n}] \) of the futures-spot basis can be rewritten as \( \mathbb{E}_t^{RN} [s_{t+n}] - \mathbb{E}_t^M [s_{t+n}] \).
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properly reflects any objective predictability in $s_{t+n}$ – then the true value of the regression coefficient $a_1$ is 1. Thus, if the estimated slope $\hat{a}_1$ is statistically different from 1, that suggests that futures prices are informationally inefficient and/or that futures prices include time-varying risk premia. Fama and French [87] find that the estimated slope coefficients are positive for many (but not all) commodities, but often are less than 1. These results are consistent with the existence of time-varying risk premia in futures prices. Other work on commodity risk premia is in Bessembinder [17], Bessembinder and Chan [18], and Gorton, Hayashi, and Rouwenhorst [110]. Closely related issues are the idea in Keynes [138] that commodity futures prices are affected by changing trading imbalances between speculators and hedgers (see Dewally, Ederington, and Fernanado [77] and references therein) and, more recently, the pricing impact of the financialization of commodities (see, for example, Buyuksahin, Haigh, Harris, Overdahl, and Robe [41] and Tang and Xiong [205]).

A second question concerns the behavior of commodity price randomness. Schwartz [186] and Kogan, Livdan, and Yaron [141] study the term structure of commodity futures price volatility. Pan [169] uses the risk-neutral price distribution inferred using the Breeden and Litzenberger [34] method to document time variation in commodity price volatility (and in higher moments) and shows that this time variation is correlated with factors reflecting dispersion of beliefs. Fama and French [88] investigate the effect of the macroeconomic business cycle on commodity prices. Chiarella, Kang, Nikitopoulos-Sklibosios, and To [59] investigate futures term structure volatility.

A third research question concerns the impact of model error, due to model misspecification or incorrect model calibration, on real option pricing and hedging. Secomandi, Lai, Margot, Scheller-Wolf, and Seppi [194] show that different hedging strategies – which are equivalent in the absence of model error – have large differences in their sensitivities to even small but empirically likely misspecifications of the RN futures curve dynamics.
4.6 Summary

This chapter reviewed three approaches to modeling commodity prices and valuing commodity-linked real assets and derivatives: Spot price evolution models, futures term structure models, and equilibrium asset pricing models. These models differ in their conceptual justification, complexity, and statistical fit. However, these different approaches are not mutually inconsistent. For example, each RN spot price model implies a corresponding model of futures curve dynamics and vice versa. Reduced-form models rely on replication (when commodities are assets or when the entire futures curve and its dynamics are known) to identify the RN price dynamics or simply posit RN price dynamics based on POR assumptions. Equilibrium models derive endogenous market prices via explicit assumptions about investor preferences. Within each of these three approaches, we presented a variety of specific single- and multi-factor models of commodity prices. Empirical evidence on commodity pricing was also reviewed.

4.7 Notes

General references for stochastic calculus and option valuation are Shreve [197], Hull [123], and Duffie [83]. References for each of the specific models discussed here are given in the body of this chapter. Seppi [196] gives another high level overview of commodity price modeling.
This chapter deals with the modeling of commodity storage assets. Section 5.1 introduces the business setting and summarizes the results presented in this chapter. Section 5.2 formulates the problem of managing a commodity storage asset as a Markov Decision Process (MDP). Sections 5.3 and 5.4 study the structure of an optimal operating policy for this MDP. Section 5.5 then discusses the computation of such a policy. Section 5.6 investigates the interplay between inventory trading and operational decisions. Section 5.7 concludes. Section 5.8 provides pointers to the literature.

5.1 Introduction

For a storable commodity, the economic interpretation of storage is the net amount of commodity carried over to the next period from the current period; that is, storage from the prior period plus the difference between the commodity production and consumption in the current period minus any in-kind storage losses (Williams and Wright
Professional commodity storers, hereafter referred to as merchants, trade this surplus in wholesale markets. For example, such a setting is the natural gas market at Henry Hub, Louisiana, the delivery location of the NYMEX natural gas futures contract.

Merchants need access to storage facilities to support their commodity trading activities. They may own such facilities themselves, or hold rental contracts on their capacity. In this chapter, a storage asset refers to the facility where a commodity can be physically stored, or a contractual agreement that entitles its owner to usage of a portion of such a facility. The storage technology may take many forms, from conventional warehouses for metals and tanks for petroleum products and chemicals to underground depleted reservoirs, aquifers, and salt domes for natural gas. Storage assets feature two distinctive operational characteristics: minimum/maximum inventory levels (space) and injection/withdrawal (flow) capacity limits. On the financial side, commodity prices are notoriously variable and volatile (Chapter 4), and storage assets give merchants the real option to buy the commodity at one point in time, store it, and sell it at a later point in time to exploit intertemporal price variability and volatility.

Merchant management of a commodity storage asset requires determining an inventory trading policy that, given the current commodity spot and futures prices and the inventory in the storage facility, tells the merchant how much commodity to buy from the wholesale market and inject into this facility, or to withdraw from this facility and sell into the wholesale market. This is a foundational problem in the merchant management of commodities, which has been studied in the literature on the warehouse problem. Cahn [42] introduces this problem as follows:

Given a warehouse with fixed capacity and an initial stock of a certain product, which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales?

Figure 2.1 on page 17 in Chapter 2 illustrates the time evolution of the inventory added to or removed from storage.
The fixed capacity attribute that Cahn mentions refers to a finite size warehouse without restrictions on how fast it can be filled up or emptied. In addition, storage facilities in practice often feature injection/withdrawal capacity constraints. That is, they are often capacitated in addition to having finite size. For example, this occurs in natural gas storage. As shown in this chapter, these additional flow capacity constraints are critical.

This chapter formulates the warehouse problem for a competitive merchant as a finite-horizon Markov decision process (MDP). This formulation encompasses both the fast storage asset case and the slow storage asset case: a fast storage asset features only space constraints, while a slow storage asset features both space and capacity constraints.

In each stage, the state of the MDP includes both the merchant’s available inventory and the current commodity spot price, with this price evolving as an exogenous single-factor Markov process. In general, the state space is uncountable. This MDP optimizes the market value of the inventory trading policy of a price-taking merchant subject to space and possibly capacity limits. The presence of both space and capacity limits gives rise to kinked injection/withdrawal capacity functions of inventory. This MDP is based on risk-neutral valuation (see Chapter 3).

This chapter analyzes this real option formulation to show (i) the structure of the optimal inventory trading policy, (ii) the computation of an optimal policy, and (iii) the managerial relevance of interfacing inventory trading and operation decisions. The main results and insights of this analysis are now summarized.

The structure of the optimal inventory trading policy. The optimal policy in the fast asset case has a simple BI/DN/WS structure: in each stage, given the current price, it is optimal to trade to withdraw and sell (WS) to empty the asset or to buy and inject (BI) to fill it up, or to do nothing (DN) otherwise. This inventory policy is a critical level,

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2 Modeling this price evolution with more than one factor would require including in the state the futures price term structure or more broadly other information, such as weather and operating conditions. Chapter 6 presents a stochastic dynamic programming model that includes the entire futures price term structure in its states.
or, equivalently, *basestock level* policy, in which, given a stage and a spot price, the same type of action, BI/DN/WS, is optimal independent of the inventory level. In other words, the inventory affects how much must be bought or sold to fill up or empty the asset, but not the prices at which it is optimal to buy or sell.

In the slow asset case, a modified critical level policy, also known as a *basestock target* policy, is optimal. In a given stage and for a given spot price, this policy is characterized by two BI and WS basestock targets that partition the feasible inventory set into three regions, where the BI, DN, and WS actions are respectively optimal. When a BI/WS action is optimal, it is optimal to buy-and-inject/withdraw-and-sell that amount of inventory that brings the current inventory level as close as possible to the BI/WS target. That is, these targets may not be reachable from some inventory levels. The BI/DN/WS structure is more complex than in the fast asset case, because in the fast case only one of these inventory regions is nonempty in a given stage and for a given spot price.

The BI/DN/WS structure is complex in the slow asset case because given a commodity price and a stage the type of the merchant's optimal action can depend on inventory availability. That is, at the same commodity price and stage the BI, DN, and WS actions can each be optimal at different inventory levels. This complexity is a direct consequence of the kinked capacity functions, where the kinks arise from the limited inventory adjustment capacity relative to the available space. Due to this limited capacity, the manager of a slow storage asset may at times optimally decide to make partial inventory adjustments. Put differently, this manager is concerned with managing "left over" space or inventory, that is, the available space or inventory minus the inventory adjustment capacity per stage. This issue disappears in the fast asset case in which the left over space or inventory are always zero. Thus, trading at capacity is generally suboptimal with a slow asset, while it is optimal with a fast asset. Moreover, capacity underutilization can optimally occur at *every* inventory level for which trading (BI or WS) is optimal. In contrast, optimal capacity underutilization can only occur at a trivial level with a fast asset; that is, when it is optimal to do nothing.

The optimal basestock levels/targets are functions of the stage and
the spot price. Under some assumptions on the risk-neutral dynamics of the spot price, in every stage the optimal basestock levels/targets decrease in the spot price. In every stage, the inventory and spot price space is thus partitioned into three ordered BI, DN, and WS regions.

It is also insightful to interpret the BI/DN/WS structure in the slow asset case in terms of high and low commodity prices. At a given decision stage a merchant managing a slow asset cannot always tell whether a given commodity price is high (and inventory should be sold) or low (and inventory should be bought) independently of the inventory availability. In other words, at a given time, the same price can be high or low at different inventory levels. This insight is markedly different from the inventory-independent characterization of a given spot price as high or low in the fast asset case. This result occurs even if the merchant is risk-neutral and price-taker. Thus, nonlinearities in the operations of a commodity storage asset can fundamentally condition a merchant qualification of high and low commodity prices. The relevant nonlinearities in the storage asset operations are brought about by the interplay between the space and capacity limits that give rise to kinked capacity functions in the slow asset case.

**The computation of an optimal policy.** When the spot price dynamics are modeled using a lattice representation, the basestock levels for a fast storage asset can be easily computed by solving a discrete state stochastic dynamic program (SDP) via backward dynamic programming. If in addition the operational parameters of a slow storage asset satisfy a benign technical condition, the basestock targets can be computed in an analogous manner. The ability to compute an optimal storage policy facilitates further numerical analysis.

**The managerial importance of interfacing operational and trading decisions.** The complex nature of the BI/DN/WS structure for a slow asset implies that the interface between inventory trading and operational decisions is in general nontrivial. These choices cannot be optimally decoupled by letting the inventory trader decide at what price to buy/sell (that is, set the BI and WS prices) as if the asset were fast and then having the operations manager inject/withdraw at capacity.
This chapter uses natural gas data to quantify the importance of coupling these decisions.\(^3\) It is shown here that mismanaging the trading and operations interface can yield substantial value losses. Moreover, incorrect inventory trading decisions magnify these losses when the injection capacity is increased. A surprising part of these results is that they hold even when the addition of constraining injection/withdrawal capacity does not dramatically reduce the value of an optimally managed storage asset, which is due to the fact that in these cases the asset is operated at similar flow rates. Thus, nonlinear capacity functions make managing a commodity storage asset a genuinely difficult problem.

The results and insights of this chapter have relevance for the management of natural gas storage assets, as well as of storage assets for other (partially) storable commodities, such as oil, metals, and agricultural products (see §2.1 in Chapter 2). While the facilities used to store these commodities may feature less stringent engineering restrictions in terms of inventory adjustment capacities, in such cases capacity limitations equivalent to those studied in this paper can arise as a consequence of logistical and market constraints that limit how fast a merchant can effectively fill up or empty a facility.

### 5.2 Model

This section describes a periodic review model where inventory trading decisions are made at \(N\) given equally spaced points in time. Set \(\mathcal{I} := \{0, \ldots, N - 1\}\) indexes them; that is, the \(i\)-th decision, \(i \in \mathcal{I}\), is made at time \(T_i\). An action is denoted as \(a\): a positive action corresponds to a purchase followed by an injection, a negative action to a withdrawal followed by a sale, and zero is the do-nothing action. An

\(^3\)This analysis is not based on the premise that policies used in practice to manage slow commodity storage assets decouple inventory trading and operational choices. Indeed, the practice-based policies benchmarked in Chapter 6 do consider the capacity constraints of slow storage assets. Hence, the goal of the analysis conducted in §5.6 is not to estimate the likely suboptimality of losses actually incurred in practice, rather it is to understand in a realistic setting the importance of considering the capacity constraints when managing slow storage assets.
injection or a withdrawal corresponding to a decision made at time \( T_i \) is executed as a flow during the time interval in between times \( T_i \) and \( T_{i+1} \). This means that commodity purchased/sold at time \( T_i \) is available/unavailable in storage at time \( T_{i+1} \). For the most part, a buy-and-inject or withdraw-and-sell action will be simply referred to as an injection or a withdrawal, or, more generally, a trade. The monetary payoff of the \( i \)-th trade occurs at time \( T_i \). This modeling timing reflects typical practice where financial payoffs are accounted for at specific points in time, even though physical operations often occur as flows over time. Moreover, if the length of the review period is set sufficiently short, this modeling set-up can closely approximate financial payoffs that occur simultaneously with the operational execution of the trades.

The storage asset has minimum and maximum inventory levels, \( \underline{x} \) and \( \bar{x} \in \mathbb{R}_+ \), with \( \underline{x} < \bar{x} \) (\( \bar{x} > 0 \) is common in energy applications). Hence, the feasible inventory set is \( X := [\underline{x}, \bar{x}] \). There are constant injection and withdrawal capacities \( C^I > 0 \) and \( C^W < 0 \), respectively, on the maximum amount of inventory that can be injected into and withdrawn out of the facility in each review period (to be strict this applies to the absolute value of \( C^W \)). It is assumed that both \( C^I \) and \( -C^W \) belong to the set \( (0, \bar{x} - \underline{x}] \). This implies that the storage asset features inventory-dependent injection and withdrawal capacity functions of inventory \( \pi(x) : X \to [0, C^I \wedge (\bar{x} - \underline{x})] \) and \( g(x) : X \to [C^W \vee (\underline{x} - \bar{x}), 0] \) defined as

\[
\begin{align*}
\pi(x) & := C^I \wedge (\bar{x} - x), \\
g(x) & := C^W \vee (\underline{x} - x),
\end{align*}
\]

(5.1)  (5.2)

where \( \cdot \wedge \cdot \equiv \min\{\cdot, \cdot\} \) and \( \cdot \vee \cdot \equiv \max\{\cdot, \cdot\} \). These functions express the maximum amount of commodity that can be injected and withdrawn, respectively, into and out of the facility during each review period starting from inventory level \( x \in X \) while keeping the inventory level in set \( X \) (strictly speaking this comment applies to the absolute value of \( g(x) \)). The fast asset case arises when \( C^I = -C^W = \bar{x} - \underline{x} \).

Figure 5.1 illustrates the capacity functions in these cases. In the slow facility case, the kinks in these functions (at \( x = \bar{x} - C^I \) and \( x = \underline{x} - C^W \) in the injection and withdrawal cases, respectively) play a
fundamental role in the analysis of the structure of the optimal trading policy carried out in §5.3. These kinks are not present in the fast facility case.

At every review time, the sets of feasible withdrawal and injection actions, respectively, with inventory level \( x \in \mathcal{X} \) are \( \mathcal{A}^W(x) := [a(x), 0] \) and \( \mathcal{A}^I(x) := [0, \overline{a}(x)] \), and the set of all feasible actions is \( \mathcal{A}(x) := \mathcal{A}^W(x) \cup \mathcal{A}^I(x) \). Figure 5.1 also illustrates the feasible inventory action set \( \mathcal{C} := \{(x, a) : x \in \mathcal{X}, a \in \mathcal{A}(x)\} \), both in the fast and slow cases.

Let the commodity spot price at time \( T_i \) be the random variable \( \tilde{s}_i \), with \( \mathcal{S}_i \subseteq \mathbb{R}_+ \) the set of its possible realizations (for notational simplicity \( T_i \) is abbreviated to \( i \) when subscripting other variables).\(^4\)

\(^4\)The possibility of negative prices, which can occur in some electricity markets, is not considered here. Section §5.8 discusses papers that present models that admit negative
A Markovian stochastic process \( \{ \tilde{s}_i, i \in I \} \) models the evolution of the spot price, starting from the given initial price \( s_0 \) at the initial date \( T_0 \). Assuming a single-factor Markovian model of the spot price evolution, the spot price is a sufficient statistic for the spot price evolution. The spot price evolves independently of the merchant trading decisions, which is consistent with the assumption of a price-taking merchant. Each merchant trade here is small relative to the size of the commodity market.

A trading decision \( a \) at time \( T_i, i \in I \), depends on the realized spot price, \( s \in S_i \), and, by the restriction \( a \in A(x) \), on the available inventory, \( x \), at this time. The immediate reward (cash flow) associated with this decision is \( r(a, s) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \). Let \( \phi^W \in (0, 1] \) and \( \phi^I \geq 1 \) denote commodity price adjustment factors used to model in-kind fuel costs that arise in the context of natural gas storage (e.g., fuel burned by compressors). Let \( c^W \) and \( c^I \) be positive constant marginal withdrawal and injection costs, respectively. Define the buy-and-inject adjusted price and the withdraw-and-sell adjusted price as \( s^I := \phi^I s + c^I \) and

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electricity prices.
5.2. Model

The immediate reward function is

\[ r(a, s) := \begin{cases}  -s^W a, & \text{if } a \in \mathbb{R}_-, \\ 0, & \text{if } a = 0, \\ -s^I a, & \text{if } a \in \mathbb{R}_+ , \end{cases} \quad \forall s \in \mathbb{R}_+ . \]

Figure 5.2 illustrates this function (while not shown in this figure, this function is not necessarily positive in a withdrawal action). The reward function is kinked in the action at zero, as \( s^W \leq s^I \). In contrast to the kinks in the capacity functions, the reward function kink does not play an important role in the analysis of §5.3, in the sense that the structural characterization of the optimal policy would persist even without this kink (that is, if the immediate payoff function were linear in the action).

A unit cost \( h \) for physically holding (but not financing) inventory is charged at each time \( T_i \), \( i \in \mathcal{I} \), against the inventory \( x \in \mathcal{X} \) available at this time. Cash flows are discounted from time \( T_i \) back to time \( T_{i-1} \), \( i \in \mathcal{I} \setminus \{0\} \), using the deterministic risk-free discount factor \( \delta_{i-1} \in (0, 1] \).

An optimal inventory trading policy can be obtained by solving a finite horizon MDP. Set \( \mathcal{I} \) indexes the stages and the state space in stage \( i \) is \( \mathcal{X} \times \mathcal{S}_i \). Let \( A^\pi_i(x, s) \) be the decision rule of policy \( \pi \) in stage \( i \). Denote by \( \Pi \) the set of all the feasible policies. Let \( \tilde{x}^\pi_i \) be the random inventory level available in stage \( i \) when following policy \( \pi \). The objective is to find an optimal policy for the following optimization model:

\[
\max_{\pi \in \Pi} \sum_{i=0}^{N-1} \left( \prod_{j=0}^{i} \delta_j \right) \mathbb{E}^{\text{RN}} \left[ r \left( A^\pi_i \left( \tilde{x}^\pi_i, \tilde{s}_i \right), \tilde{s}_i \right) - h \tilde{x}^\pi_i \mid x_0, s_0 \right], \tag{5.3}
\]

where \( \mathbb{E}^{\text{RN}}[\cdot \mid x_0, s_0] \) denotes risk-neutral expectation given the initial inventory \( x_0 \) and spot price \( s_0 \).

Model (5.3) can be formulated as an SDP. Such a reformulation is useful both for deriving the structure of an optimal policy and computing such a policy. Denote by \( V_i(x, s) \) the optimal value function in stage \( i \) and state \( (x, s) \). The SDP formulation of model (5.3) is

\[
V_{N-1}(x, s) := \max_{a \in A^W(x)} -s^W a - h x, \quad \forall (x, s) \in \mathcal{X} \times \mathcal{S}_i, \tag{5.4}
\]

\[
V_i(x, s) = \max_{a \in A(x)} v_i(x, a, s), \quad \forall i \in \mathcal{I} \setminus \{N - 1\}, \quad (x, s) \in \mathcal{X} \times \mathcal{S}_i,
\]
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\begin{equation}
 v_i(x,a,s) := r(a,s) - hx + \delta_i \mathbb{E}_i^{RN} \left[ V_{i+1}(x + a, \tilde{s}_{i+1}) \right],
\end{equation}

\begin{equation}
 \text{where } \mathbb{E}_i^{RN} \left[ \cdot \mid \tilde{s}_i = s \right] \text{ is the time } T_i \text{ risk-neutral conditional expectation given that } \tilde{s}_i \text{ is equal to the spot price } s \in S_i, \forall i \in \mathcal{I}. \text{ This formulation is interpreted as follows. In the last stage, only the WS and DN actions are available. In prior stages, the BI, DN, and WS actions are available.}

\text{In each stage and state, one action is performed. The immediate reward function } r(a,s) \text{ is superadditive with respect to two actions with opposite signs for each given spot price; that is, given } a^W < 0 < a^I \text{ it holds that } r(a^W,s) + r(a^I,s) \leq r(a^W + a^I,s), \forall i \in \mathcal{I}, s \in S_i. \text{ This property implies that it is never optimal to perform two opposite trades in the same stage, so that the SDP formulation (5.4)-(5.6) is without loss of generality.}

5.3 Basestock Optimality

This section analyzes the structure of an optimal inventory trading policy given the spot price and stage. The following natural assumption on the risk-neutral expected spot price holds throughout. It ensures that the value function is finite in every stage and state.

**Assumption 1 (Finite risk-neutral expected spot price).** It holds that \( \mathbb{E}^{RN}[\tilde{s}_j | \tilde{s}_i = s] < \infty, \forall i \in \mathcal{I} \setminus \{N - 1\}, j \in \mathcal{I}, \text{ with } j > i, \text{ and } s \in S_i. \)

In the ensuing analysis, properties of functions and comparisons should be interpreted in the weak sense. It is useful to define the continuation value function:

\begin{align*}
 W_{N-1}(x,s) & := 0, \ \forall (x,s) \in \mathcal{X} \times S_{N-1}, \\
 W_i(x,s) & := \delta_i \mathbb{E}_i^{RN} \left[ V_{i+1}(x, \tilde{s}_{i+1}) \mid \tilde{s}_i = s \right], \\
 & \forall i \in \mathcal{I} \setminus \{N - 1\}, (x,s) \in \mathcal{X} \times S_i.
\end{align*}
5.3. Basestock Optimality

5.3.1 Fast Asset

In the case of a fast asset, Proposition 1 provides the structure of the value and continuation value functions for a given spot price and stage.

Proposition 1 (Linearity). The value and continuation value functions $V_i(x,s)$ and $W_i(x,s)$ are linear in inventory $x \in X$ in every stage $i \in I$ and for each given spot price $s \in S_i$.

Proposition 1 implies that in every stage $i$ and for each given spot price $s$ the marginal value of inventory – that is, the slope of the value function, $V_i(x,s)$, with respect to inventory – is a constant, denoted by $\zeta_i(s)$. As the storage asset can be completely filled up or emptied in a single stage, the impact on the value of the asset of every additional unit of inventory is the same. Moreover, Proposition 1 implies that in every stage $i$ and for each given spot price $s$ the current expected next stage marginal value of inventory, that is, the slope of the continuation value function, $W_i(x,s)$, with respect to inventory, is a constant, denoted by $\overline{\zeta}_i(s)$, which satisfies $\overline{\zeta}_i(s) \equiv \delta E^{RN}[\tilde{s}_{i+1}|\tilde{s}_i = s]$ for every stage $i \in I \setminus \{N - 1\}$.

The quantity $\overline{\zeta}_i(s)$ is what Charnes et al. [54] refer to as the inventory evaluator. This quantity plays an important role in characterizing the structure of an optimal policy for a fast asset, as stated in Theorem 5.1. An optimal action in stage $i$ and state $(x, s)$ is denoted as $a_i^*(x, s)$, and the quantities $\underline{b}_i(s)$ and $\overline{b}_i(s)$ are the basestock levels for stage $i$ and spot price $s$.

Theorem 5.1 (Basestock levels). In every stage $i \in I$, given the spot price $s \in S_i$, the basestock levels and optimal action are (BI) $\underline{b}_i(s) = \overline{b}_i(s) = \overline{x}$ and $a_i^*(x, s) = \overline{x} - x$ if $s^I < \overline{\zeta}_i(s)$; (DN) $\underline{b}_i(s) = x$, $\overline{b}_i(s) = \overline{x}$, and $a_i^*(x, s) = 0$ if $s^W \leq \overline{\zeta}_i(s) \leq s^I$; and (WS) $\underline{b}_i(s) = \overline{b}_i(s) = \overline{x}$ and $a_i^*(x, s) = \overline{x} - x$ if $\overline{\zeta}_i(s) < s^W$.

Figure 5.3 illustrates the result stated in Theorem 5.1 for a given stage and spot price. The determination of an optimal action clearly involves comparing the inventory evaluator, $\overline{\zeta}_i(s)$, against the BI or
WS  spot  prices,  $s^I$  or  $s^W$,  associated  with  the  spot  price  $s$.  As  the  inventory  evaluator  does  not  depend  on  the  inventory  level,  the  same  type  of  action  is  optimal  at  all  inventory  levels.  Consequently,  it  is  optimal  to  purchase  inventory  to  fill  the  asset  up  to  inventory  level  $\bar{x}$  if  the  BI  spot  price  is  less  than  the  inventory  evaluator,  withdraw  inventory  to  empty  the  asset  down  to  inventory  level  $\underline{x}$  if  the  WS  spot  price  exceeds  the  inventory  evaluator,  and  do  nothing  otherwise.

This  is  a  simple  BI/DN/WS  structure,  in  which  if  buying-and-injection  is  optimal  the  basestock  levels  $\underline{b}_i(s)$  and  $\overline{b}_i(s)$  are  both  equal  to  $\bar{x}$,  if  withdrawing-and-selling  is  optimal  they  are  both  equal  to  $\underline{x}$,  and  if  doing  nothing  is  optimal  then  the  basestock  level  $\underline{b}_i(s)$  is  equal  to  $\underline{x}$  and  the  basestock  level  $\overline{b}_i(s)$  is  equal  to  $\bar{x}$.

5.3.2 Slow Asset

The  analysis  of  the  slow  asset  case  generalizes  the  structural  results  for  the  fast  asset.  Proposition  2  gives  a  basic  property  of  the  value  and  continuation  value  functions  for  slow  assets.

**Proposition 2 (Concavity).**  In  every  stage  $i \in I$,  the  value  function  $V_i(x, s)$  and  the  continuation  value  function  $W_i(x, s)$  are  concave  in
inventory $x \in \mathcal{X}$ for each given spot price $s \in \mathcal{S}_i$.

In contrast to the fast case, Proposition 2 implies that the marginal value of inventory and the inventory evaluator are no longer constant given a stage and a spot price. That is, these quantities are inventory-dependent. Due to the limited injection and/or withdrawal capacities, every additional unit of inventory has a smaller impact on the asset value when inventory increases. This difference between the fast and slow asset cases directly affects the basestock levels, which, as stated in Theorem 5.2, now become basestock *targets*, which may not be reachable from some inventory levels.

**Theorem 5.2 (Basestock targets).** In every stage $i \in \mathcal{I}$ and for each given spot price $s \in \mathcal{S}_i$, there exist basestock targets $\underline{b}_i(s)$ and $\overline{b}_i(s) \in \mathcal{X}$, with $\underline{b}_i(s) \leq \overline{b}_i(s)$, such that an optimal action in each state $(x, s) \in \mathcal{X} \times \mathcal{S}_i$ is

$$ a_i^*(x, s) = \begin{cases} 
(\overline{b}_i(s) - x) \wedge C_I, & \text{if } x \in [\underline{b}_i(s), \overline{b}_i(s)], \\
0, & \text{if } x \in (\underline{b}_i(s), \overline{b}_i(s)], \\
(\overline{b}_i(s) - x) \vee C_W, & \text{if } x \in (\overline{b}_i(s), \bar{x}]. 
\end{cases} $$

Figure 5.4 illustrates the intuition behind the basestock target result in Theorem 5.2 for a given stage and spot price. The solid line segments in this figure are the discounted risk-neutral expected next-stage marginal value of inventory, which corresponds to the left derivative with respect to inventory, $\zeta_i(x, s)$, of the continuation value function, $W_i(x, s)$. In contrast to the fast asset, the expected next stage marginal value of inventory now depends on the inventory level, and hence the extension of the notation $\zeta_i(s)$ to $\zeta_i(x, s)$. The function $\zeta_i(x, s)$ is defined as

$$ \zeta_i(x, s) := \lim_{\varepsilon \downarrow 0} \frac{W_i(x, s) - W_i(x - \varepsilon, s)}{\varepsilon}, $$

for all $(i, x, s) \in \mathcal{I} \setminus \{N - 1\} \times X^o \times \mathcal{S}_i$, where $X^o$ is the interior of $\mathcal{X}$. This definition is extended to inventory levels $\underline{x}$ and $\bar{x}$ as follows:

$$ \zeta_i(\underline{x}, s) := \lim_{x \downarrow \underline{x}} \zeta_i(x, s), $$

$$ \zeta_i(\bar{x}, s) := \lim_{x \uparrow \bar{x}} \zeta_i(x, s), $$
for all \((i, s) \in \mathcal{I} \setminus \{N - 1\} \times \mathcal{S}_i\). With these conventions, \(\bar{\zeta}_i(x, s)\) is defined for all \(x \in \mathcal{X}\) for each \(s \in \mathcal{S}_i\).

Figure 5.4 assumes that the continuation value function is piecewise linear concave in inventory (Section 5.5 provides conditions for this to be the case). As in the fast case, a BI action or a WS action is optimal, respectively, if the BI spot price \(s^I_i\) or the WS spot price \(s^W_i\) is below or above the discounted risk-neutral expected next-stage marginal value of inventory, \(\bar{\zeta}_i(x, s)\), and the DN action is optimal otherwise. However, given the spot price, since the discounted risk-neutral expected next-stage marginal value of inventory decreases in inventory, this function brackets the BI spot price and the WS spot price at no more than two critical inventory levels. These critical inventory levels are the basestock targets \(b^I_i(s)\) and \(b^W_i(s)\). These basestock targets are ordered, that is, \(b^I_i(s) \leq b^W_i(s)\), because the WS spot price is less than the BI spot price. These targets thus partition the feasible inventory set into no more than three contiguous regions, where BI, DN, and WS actions are respectively optimal. These regions are the sets \([\bar{x}, b^I_i(s)]\), \([b^I_i(s), \bar{b}^I_i(s)]\), and \((\bar{b}^I_i(s), \bar{x})\).

This is an extended version of the BI/DN/WS structure that holds for the fast asset case. With a slow asset, the basestock targets in a
5.3. Basestock Optimality

given stage and for a given spot price are not necessarily equal to the minimum or maximum inventory levels. This is why they are targets rather than levels.

These observations yield the following insights. First, in the slow asset case, at the same spot price any type of action can be optimal in a given stage depending on the available inventory. Thus, in general, it is impossible to provide an absolute characterization of a spot price in a given stage as low or high; that is, one at which a BI action or a WS action is optimal, respectively. Any such statement must be made relative to inventory availability. In contrast, in the fast facility case in a given stage it is indeed possible to define a price as low, intermediate (that is, the DN action is optimal at this price), and high independently of the available inventory.

Second, in the fast asset case, when trading (a BI action or a WS action) is optimal, it is optimal to fully utilize the capacity functions. An optimal BI action is equal to the value taken by the injection capacity function (5.1) and an optimal WS action is equal to the value taken by the withdrawal capacity function (5.2). In contrast, in the slow asset case, when trading is optimal, it can be optimal to underutilize the available capacity, as expressed by the capacity functions, at some inventory levels. What may be less clear is that this capacity underutilization can occur at every inventory level for which trading is optimal. Example 1 illustrates this possibility.

Example 1 (Slow injection capacity). In this example there are three time periods ($N = 3$). To emphasize the role played by the capacity functions (5.1)-(5.2), the injection/withdrawal marginal costs and the holding cost are set to 0 and the price adjustment factors and the discount factor are equal to 1. Thus, it holds that $r(a, s) = -sa$ at each stage. Prices are deterministic. As illustrated in Figure 5.5, the path of prices is $s_0 = s^M$, $s_1 = s^L$, and $s_2 = s^H$, where $s^H > s^M > s^L > 0$ and the superscripts $H$, $M$, and $L$ abbreviate high, medium, and low, respectively.

With a fast asset, clearly, one would fill up the optimally empty asset in stage 1 at the low price $s^L$, and sell the entire inventory (down to $\underline{x}$) in stage 2, at the high price $s^H$. 
Now consider a storage asset that can be emptied in one period (fast withdrawal capacity function), while filling it up requires more than one period but less than two periods (slow injection capacity function). With an empty asset, in stage 0 it is optimal to buy inventory and partially fill up the asset, hence optimally underutilizing the injection capacity at each inventory level where a BI trade is optimal. With a full asset, in stage 0 it is optimal to partially withdraw-and-sell the available inventory, hence optimally underutilizing the withdrawal capacity. In both cases, in stage 1 it is optimal to buy additional inventory and completely fill up the asset. This basic intuition is now formalized.

Since, when empty, the asset can be filled up in less than two periods but in more than one period, it holds that \( x + 2C^I > \bar{x} > x + C^I \), which can be rewritten as \( C^I < \bar{x} - x < 2C^I \).

In stage 2, it holds that \( V_2(x, s^H) = s^H(x - x) \) because \( A^W(x) = [\bar{x} - x, 0] \), \( \forall x \in \mathcal{X} \). In stage 1, with \( x \in \mathcal{X} \) and \( a \in [\bar{x} - x, C^I \wedge (\bar{x} - x)] \), it is easy to verify that, from (5.6), \( v_1(x, a, s^L) = (s^H - s^L)a + s^H(x - x) \).

Since \( v_1(x, a, s^L) \) increases in \( a \), it follows that \( a^*_1(x, s^L) = C^I \wedge (\bar{x} - x) \) and \( V_1(x, s^L) = (s^H - s^L)[C^I \wedge (\bar{x} - x)] + s^H(x - x) \), \( \forall x \in \mathcal{X} \). Finally, consider stage zero and focus on inventory level \( \underline{x} \). For \( a \in [0, \bar{x} - x] \),
it holds that $v_0(\bar{x}, a, s^M) = (s^H - s^M)a + (s^H - s^L)[C^I \land (\bar{x} - \bar{x} - a)]$.

Since

$$C^I \land (\bar{x} - \bar{x} - a) = \begin{cases} C^I, & \text{if } a \in [0, \bar{x} - \bar{x} - C^I), \\ \bar{x} - \bar{x} - a, & \text{if } a \in [\bar{x} - \bar{x} - C^I, \bar{x} - \bar{x}], \end{cases}$$
it follows that

\[ v_0(\bar{x}, a, s^M) = \begin{cases} 
  (s^H - s^M)a + (s^H - s^L)C^I, & \text{if } a \in [0, \bar{x} - C^I), \\
  (s^L - s^M)a + (s^H - s^L)(\bar{x} - \bar{s}), & \text{if } a \in [\bar{x} - C^I, \bar{x} - \bar{s}].
\end{cases} \]

It holds that \( \bar{x} - C^I =: a_0(\bar{x}, s^M) \in \arg \max_{a \in [0, \bar{x} - \bar{s}]} v_0(\bar{x}, a, s^M) \) and, by (5.7), \( 0 < a_0(\bar{x}, s^M) < C^I = \bar{a}(\bar{x}) \). Thus, in stage 0 at inventory level \( \bar{x} \) it is optimal to buy and inject without fully utilizing the injection capacity. Moreover, it holds that \( g(\bar{x}) = a_0(\bar{x}, s^M) \in \arg \max_{a \in [\bar{x} - C^I, 0]} v_0(\bar{x}, a, s^M) \). Thus, in stage 0 at inventory level \( \bar{x} \) it is optimal to withdraw and sell without fully utilizing the withdrawal capacity. Since \( \bar{x} + a_0(\bar{x}, s^M) = \bar{x} - C^I \) and \( \bar{x} + a_0(\bar{x}, s^M) = \bar{x} - C^I \), it holds that \( b_0(s^M) = b_0(s^M) = \bar{x} - C^I \) and \( a_0(x, s^M) = \bar{x} - x - C^I, \) \( \forall x \in X \). Thus, in stage 0 it is optimal to buy and inject without fully utilizing the injection capacity at every inventory level in the interval \([\bar{x}, \bar{x} - C^I]\), and it is optimal to withdraw and sell without fully utilizing the withdrawal capacity at every inventory level in the interval \((\bar{x} - C^I, \bar{x}]\). Figure 5.6 displays the behavior of this optimal action as a function of inventory and relates it to those of the injection and withdrawal capacity functions.

This example illustrates a general feature of an optimal basestock target structure for a slow storage asset: limited inventory adjustment capacity forces the manager of a storage asset to partially shift the decision to adjust the inventory level in a given stage to stages with less attractive cash flows. In other words, the complexity of an optimal basestock target structure arises from the need to manage “left over” space or inventory arising from limited capacity. For instance, in Example 1 the space left over is the difference between the available space with an initially empty asset and the injection capacity, that is, \( (\bar{x} - \bar{s}) - C^I \).

### 5.4 Price Monotonicity of the Basestock Targets

This section characterizes the behavior of the basestock targets in the spot price in a given stage. For simplicity of exposition, the ensuing analysis only uses the basestock target terminology, but this analy-
5.4. Price Monotonicity of the Basestock Targets

Table 5.1 BI and WS unit margins, basestock targets, and optimal action type in stage 0 in Example 2.

<table>
<thead>
<tr>
<th>Function</th>
<th>0.01</th>
<th>2</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{s}_{1</td>
<td>0}(s) - (s + 0.04) )</td>
<td>-0.0351</td>
<td>0.1522</td>
<td>-0.0025</td>
</tr>
<tr>
<td>( \bar{s}_{1</td>
<td>0}(s) - s )</td>
<td>0.0049</td>
<td>0.1922</td>
<td>0.0375</td>
</tr>
<tr>
<td>( \bar{b}_0(s) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \bar{b}_0(s) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Optimal action type | DN   | BI   | DN   | WS   |

s is also applies to the fast asset case, that is, when these targets are basestock levels reachable from any inventory level.

In a given stage, the basestock targets are said to be monotonic in the spot price if they decrease in the spot price. Example 2 shows that a basestock target does not always satisfy this property.

Example 2 (Nonmonotonic basestock target). Let \( N = 2, \bar{x} = 0, \) and \( \bar{x} = 1 \). Suppose that \( \phi^I = \phi^W = 1, c^I = 0.04, c^W = 0, \delta_1 = 1, \) \( h = 0, \) and \( C^I = -C^W = 1 \). This is a fast asset, so that \( V_1(x, s) = x s \) and \( W_0(x, s) = x \bar{s}_{1|0}(s) \), where, with a slight abuse of notation, \( \bar{s}_{1|0}(s) := \mathbb{E}^{RN}[\bar{s}_1|\bar{s}_0 = s] \). Assume that there are four possible spot prices in stage 0, those in set \( S_0 = \{0.01, 2, 9, 12\} \), and that \( \bar{s}_{1|0}(s) \) is equal to 0.0149, 2.1922, 9.0375, and 11.8498 for \( s = 0.01, 2, 9, \) and 12, respectively. In stage 0, \( b_0(s) \) and \( \bar{b}_0(s) \) are optimal solutions to

\[
\max_{y \in [0,1]} [\bar{s}_{1|0}(s) - (s + 0.04)]y
\]

and

\[
\max_{y \in [0,1]} [\bar{s}_{1|0}(s) - s]y,
\]

respectively. Thus, \( b_0(s) \) is equal to 0 if the BI unit margin, \( \bar{s}_{1|0}(s) - (s + 0.04) \), is negative and equal to 1 if this quantity is positive; and \( \bar{b}_0(s) \) is equal to 0 if the WS unit margin, \( \bar{s}_{1|0}(s) - s \), is negative and equal to 1 if this quantity is positive. Table 5.1 displays the BI and WS unit margins and the resulting basestock targets and optimal action types for all possible spot prices in stage 0. This table shows that \( b_0(s) \) behaves nonmonotonically in \( s \).

What makes the function \( b_0(s) \) nonmonotonic in \( s \) in Example 2 is the nonmonotonic behavior of the function \( \bar{s}_{1|0}(s) - s \) in \( s \). In this example, had the latter function decreased in \( s \), the function \( b_0(s) \) would...
have been monotonic in \( s \). This observation suggests that the price monotonicity of the basestock targets can be established by imposing more structure on the spot price process. The analysis in this section imposes on the spot price process the conditions stated in Assumption 2.

**Assumption 2 (Spot price process).** For every \( i \in I \setminus \{ N - 1 \} \),

(a) the distribution function of random variable \( \tilde{s}_{i+1} \) conditional on the spot price \( s \) in stage \( i \) stochastically increases in \( s \in S_i \); (b) the function \( \delta_i \phi_i \mathbb{E}_i^R [\tilde{s}_{i+1} | \tilde{s}_i = s] - \phi_W s \) decreases in \( s \in S_i \).

The condition stated in part (a) of Assumption 2 is rather natural. It implies that the risk-neutral expected spot price in the next stage increases in the current spot price (see, e.g., Topkis [215, Corollary 3.9.1(a)]). Part (b) of Assumption 2 is motivated by the observation made after Example 2. If both \( \phi_i^f \) and \( \phi_W \) are equal to 1, it implies that the discounted risk-neutral expected spot price in the next stage, given the spot price in the current stage, increases at a slower rate than this spot price. The Jaillet-Ronn-Tompaidis (mean-reverting) model (see §4.2 in Chapter 4), used in §5.6, satisfies part (a) of Assumption 2 by Theorem 4 in Müller [163], but not part (b). (Indeed, the risk-neutral expected spot prices in stage 1 conditional on the stage 0 spot prices used in Example 2 are obtained by using a mean-reverting model.) A model that satisfies both of these conditions is Geometric Brownian Motion with appropriate drift rate. The Schwartz-Smith model (see §4.2 in Chapter 4) uses Geometric Brownian Motion to represent long-term variations in commodity prices, and Schwartz and Smith [187] suggest that it may be an appropriate model for the valuation of long-term real options. However, part (b) of Assumption 2 is not necessary for the optimal basestock targets to be monotonic in price, as it is easy to verify that \( \bar{b}_0(s) \) is monotonic in \( s \) in Example 2 when \( c_i^f \) is zero.

Theorem 5.3 formalizes the price monotonicity of the basestock targets when Assumption 2 holds.

**Theorem 5.3 (Price monotonicity).** If Assumption 2 holds, then in every stage \( i \in I \) the basestock targets \( \bar{b}_i(s) \) and \( \tilde{b}_i(s) \) are decreasing
functions of the spot price \( s \in \mathcal{S}_i \).

To appreciate this result, focus on the slow asset case and Figure 5.4 (an analogous argument holds in the fast asset case). As the spot price \( s \) increases, both the BI spot price, \( s^I \), and the WS spot price, \( s^W \), increase and, under Assumption 2, it is possible to show that the discounted risk-neutral expected next-stage marginal value of inventory, that is, the function \( \bar{\zeta}_i(y, \cdot) \), also increases. It is thus not obvious that the optimal basestock targets decrease in the spot price. However, this would occur if the functions \( \bar{\zeta}_i(y, s) - s^I \) and \( \bar{\zeta}_i(y, s) - s^W \) decreased in the spot price \( s \) (for each given stage \( i \) and inventory level \( y \)), which is the case under Assumption 2. This is the main idea that can be used to establish Theorem 5.3.

In words, Theorem 5.3 follows from showing that the discounted risk-neutral expected next-stage marginal value of inventory net of its marginal acquisition cost and marginal disposal “cost” decreases in the spot price. Consequently, the basestock targets decrease in the spot price, or, equivalently, the optimal amount of inventory bought and injected (respectively, withdrawn and sold) in each given stage decreases (respectively, increases) in the spot price. This behavior is consistent with the discussion of complementarity in Topkis [215, pp. 92-93].

When Assumption 2 holds, Theorem 5.3 brings to light the structure of the optimal policy illustrated in Figure 5.7: in every stage \( i \in \mathcal{I} \), the basestock targets \( b_i(s) \) and \( \bar{b}_i(s) \) partition the state space \( \mathcal{X} \times \mathcal{S}_i \) into disjoint BI, DN, and WS regions. How these functions decrease in the spot price can differ significantly in the fast and slow asset cases, as shown in panels (a) and (b) of Figure 5.7.

### 5.5 Computation

Consider a Markov process for the spot price that in each stage can only take on a finite number of values, for example using a lattice model such as in Jaillet et al. [125] (see also Luenberger [155, Chapter 15] and Hull [121, Chapter 16]). Thus, in this section, the spot price is assumed to evolve as stated in Assumption 3.
Assumption 3 (Finite spot price sets). In every stage $i \in \mathcal{I}$, the spot price set $\mathcal{S}_i$ is finite.

This assumption implies that the random next-stage price $\tilde{s}_{i+1}$ conditional on each spot price $s \in \mathcal{S}_i$ has a discrete probability distribution, $\forall i \in \mathcal{I} \setminus \{N - 1\}$.

Under Assumption 3, an optimal policy in the fast asset case can be computed by dynamic programming backward recursion, despite the feasible inventory set $\mathcal{X}$ being uncountable. This is so because determining the inventory evaluator in every stage and for each given spot price only requires computing the continuation value function at the two inventory levels $\underline{x}$ and $\overline{x}$.

The remainder of this section focuses on the slow asset case. Proposition 3 gives a useful property of the value function under Assumption 3.
Proposition 3 (Piecewise linearity). If Assumption 3 holds, then, in every stage \( i \in I \), the optimal value function \( V_i(x, s) \) of a slow storage asset is piecewise linear and continuous in \( x \in \mathcal{X} \) for each \( s \in S_i \).

Under Assumption 3, computing the basestock targets for a slow asset is simplified if the capacities, \(-C^W\) and \(C^I\), and the inventory limits, \(\underline{x}\) and \(\overline{x}\), are integer multiples of some real number. In this case, Proposition 4, the proof of which uses the property given in Proposition 3, states two useful properties of the value function and basestock targets.

Proposition 4 (Restricted capacities and inventory limits). Suppose that Assumption 3 holds and that there exists a maximal number \( Q \in \mathbb{R}_+ \) such that \(-C^W, C^I, \underline{x}, \overline{x}\) are all integer multiples of \( Q \). In every stage \( i \in I \) and for each given spot price \( s \in S_i \), (a) the value function \( V_i(x, s) \) can change slope in inventory \( x \) only at inventory levels that are integer multiples of \( Q \), and (b) the basestock targets \( b_i(s) \) and \( \overline{b}_i(s) \) can be taken to be integer multiples of \( Q \).

The properties stated in Proposition 4 are useful because one can compute the basestock targets in every stage for each spot price by restricting attention to a finite number of feasible inventory levels, namely those that are multiples of the quantity \( Q \), which can be interpreted as a lot size. Thus, one needs to compute the value function in each stage and for each possible spot price only for the \( 1+(\overline{x}-\underline{x})/Q \) feasible inventory levels \( \underline{x}, \underline{x}+Q, \underline{x}+2Q, \ldots, \overline{x} \). This computation can be easily carried out by optimally solving a discrete space finite horizon SDP by standard backward recursion.

5.6 The Value of Optimally Interfacing Operational and Trading Decisions

This section quantifies the managerial relevance of the BI/DN/WS structure in the slow asset case through a computational analysis in the context of natural gas storage. The issue being investigated is the importance of taking the capacity limits into account when determin-
ing a trading policy; that is, of optimally interfacing operational and inventory trading decisions.

The analysis here assumes that the natural gas price evolves as a single-factor mean-reverting process with deterministic monthly seasonality factors as in the Jaillet-Ronn-Tompaidis model discussed in §4.2 in Chapter 4. Recall that in this model the spot price, $s_t$, at time $t$ is the exponential of a single mean-reverting factor, $\epsilon_t$, multiplied by a deterministic monthly seasonality factor, $g(t)$: $s_t = g(t) \exp(\epsilon_t)$. That is, the single mean-reverting factor, $\epsilon_t$, is the natural logarithm of the deseasonalized spot price. This factor evolves in continuous time and values according to the stochastic differential equation

$$d\epsilon_t = \alpha(\theta - \epsilon_t)dt + \sigma dZ_t,$$

where $\alpha > 0$, $\theta$, and $\sigma > 0$, respectively, are the speed of mean reversion, the long term mean-reversion level, and the volatility of $\epsilon_t$, and $dZ_t$ is an increment to a Standard Brownian Motion. The dynamics in (5.8) are under the risk-neutral measure. The dynamics of the single mean-reverting factor under the objective measure may have a different mean-reversion level.

Table 5.2 displays the estimates of the parameters of this model employed in this analysis. These estimates are obtained by calibrating the model parameters to daily NYMEX natural gas futures and option prices observed in February 2006 using the approach described by Jaillet et al. [125].

The minimum and maximum inventory levels $\underline{x}$ and $\overline{x}$ are 0 and 10, respectively. (The normalized unit of measurement of inventory should be interpreted as an appropriate mmBtu multiple, where 1 mmBtu = 1,000,000 British thermal units.) In practice, the number of days needed to fill up and empty different types of natural gas storage facilities varies considerably, between 20-250 and 10-150, respectively (FERC [90, p. 7]). The monthly injection and withdrawal capacities are thus varied from 10% to 100% of the maximum available space (10 units) in increments of 10%. This setting covers several cases of practical interest since it corresponds to varying the number of days required to fill-up/empty a facility roughly between 30 and 300.

The injection fuel loss factor is set equal to 0.01, and there is no
5.6. The Value of Optimally Interfacing Operational and Trading Decisions

Table 5.2 Estimates of the seasonal mean-reverting price model parameters based on calibrating the parameters of this model to daily NYMEX natural gas futures and option prices observed in February 2006.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed of mean reversion ($\alpha$)</td>
<td>0.7200</td>
</tr>
<tr>
<td>Long term log level ($\theta$)</td>
<td>1.9740</td>
</tr>
<tr>
<td>Volatility ($\sigma$)</td>
<td>0.6610</td>
</tr>
<tr>
<td>Monthly seasonality factors ($g(\cdot)$)</td>
<td>Estimate</td>
</tr>
<tr>
<td>January</td>
<td>1.1932</td>
</tr>
<tr>
<td>February</td>
<td>1.1948</td>
</tr>
<tr>
<td>March</td>
<td>1.1412</td>
</tr>
<tr>
<td>April</td>
<td>0.9142</td>
</tr>
<tr>
<td>May</td>
<td>0.8984</td>
</tr>
<tr>
<td>June</td>
<td>0.8998</td>
</tr>
<tr>
<td>July</td>
<td>0.9113</td>
</tr>
<tr>
<td>August</td>
<td>0.9219</td>
</tr>
<tr>
<td>September</td>
<td>0.9287</td>
</tr>
<tr>
<td>October</td>
<td>0.9412</td>
</tr>
<tr>
<td>November</td>
<td>1.0252</td>
</tr>
<tr>
<td>December</td>
<td>1.1049</td>
</tr>
</tbody>
</table>

loss associated with withdrawals, so that $\phi^I = 1/0.99$ and $\phi^W = 1$. The injection fuel loss factor accounts for the natural gas used by the facility pump to inject natural gas into the storage facility. No pumping is assumed for withdrawals. The withdrawal and injection marginal costs $c^W$ and $c^I$ are both set equal to $0.02/mmBtu. These values are realistic (Secomandi [190]). Consistent with how natural gas storage assets are leased in the U.S. the holding cost $h$ is equal to $0/mmBtu.

The analysis here features an SDP formulation with 24 stages ($N = 24$) corresponding to a monthly partition of a two-year time period. The first stage corresponds to the beginning of March and the remaining ones to the beginning of each of the next 23 months. The evolution of the spot price during these months is represented by a trinomial lattice constructed by standard methods (Jaillet et al. [125]) based on the calibrated values in Table 5.2 and the first 24 NYMEX natural
gas futures prices observed at the end of 2/1/2006 shown in Figure 5.8. Since this trinomial lattice is built using the risk-neutral measure, these initial futures prices are the conditional expectations of the spot price in each remaining stage under the risk-neutral measure given the spot price in the first stage. Here the initial price is $8.723/mmBtu, the closing price for the March 2006 contract on 2/1/2006. The trinomial lattice models the stochastic variability around the expected spot price. The monthly discount factor is 0.9958, which corresponds to an annual risk-free interest rate of 5% with continuous discounting. Each SDP is optimally solved using the structural results in Proposition 4, whose underlying assumptions are satisfied by the setting of this section.

The goal of the ensuing analysis is to understand the importance of optimally interfacing operational and inventory trading decisions. This analysis varies the “speed” of the storage asset by using different values for the injection/withdrawal capacity scale factors (ICSFs/WCSFs), defined as $C_I/\overline{x} - \underline{x}$ and $|C_W|/\overline{x} - \underline{x}$, respectively. Three policies are considered:

1. The fast capacity optimal policy (FCOP) that assumes that
ICSFs and WCSFs are both 100%.

(2) The slow capacity optimal policy (SCOP).

(3) A decoupled operations and trading policy (DOTP) that uses the FCOP buying and selling price thresholds in each stage, but where the quantities traded are restricted by the capacity constraints. In other words, this policy buys and sells whenever FCOP does so, but its actions are constrained by the capacity functions, so that it trades at capacity but it cannot always empty/fill-up the asset in a single stage. Thus, this policy mismanages the interface between operational and trading decisions in the merchant management of the storage asset. 5

The following analysis compares the relevant value functions and other quantities of interest in stage zero given the spot price in this stage and zero initial inventory. The initial value function under a given policy is referred to as the total asset value under this policy. This total value is the sum of the asset intrinsic and extrinsic values, which are defined later in this section.

It is clear that the FCOP value function exceeds that of SCOP in every stage. Of course, FCOP is only feasible if the storage asset is fast. The SCOP value function increases when more capacity is available and becomes equal to that of FCOP when ICSF and WCSF are both equal to 1.0. Figure 5.9 displays the FCOP percentage gains on the total value of the asset relative to SCOP for each relevant ICSF and WCSF combination. As expected, these gains decrease when ICSF or WCSF increase. When ICSF and WCSF are at least 0.4 and 0.5, respectively, these gains are no more than 10%. These gains increase rapidly when WCSF or ICSF decrease below 0.3. To explain these observations, Figure 5.10 displays the ratio of the FCOP and SCOP flow rates. These flow rates are computed by folding backward the expected amounts of natural gas withdrawn during each stage by FCOP and SCOP. Focusing on the withdrawn natural gas is appropriate since these policies start with no inventory and withdraw all the injected inventory by the

5 The value function of this policy can be easily computed.
last stage. That is, one would obtain the same flow rates by focusing on the amounts of injected natural gas.

The high relative value of SCOP when ICSF and WCSF are at least 0.4 and 0.5, respectively, is due to the fact that in these cases the two policies have very similar flow rates, with the FCOP flow rate (which is constant for each of the considered ICSF and WCSF combinations) being within 15% of the SCOP flow rates. It is interesting to note that in these cases the SCOP flow rates do not significantly increase by adding withdrawal capacity. Instead, the SCOP flow rates are significantly lower than the FCOP flow rate when WCSF or ICSF fall below 0.3, which explains the relatively low value of SCOP in these cases.

One might be tempted to conjecture that not much value would be lost by managing the operations and trading interface as in the DOTP; that is, trading as if this interface were not important. However, as
5.6. The Value of Optimally Interfacing Operational and Trading Decisions

Fig. 5.10 Flow rate percentage gains of FCOP relative to SCOP for different ICSFs and WCSFs; ICSF increases in the direction indicated by the displayed arrow.

shown next, this conjecture is generally incorrect.

Figure 5.11 shows the percentage total asset value losses of DOTP relative to SCOP. That a loss can exceed 100% is due to the fact that the DOTP value function can be negative. This figure illustrates that decoupling operational and trading decisions can generate significant losses in the presence of capacity constraints. These losses are relatively contained, that is, below 10%, only when WCSF is 1.0 and ICSF is 0.2 or larger, or when WCSF is 0.9 and ICSF is 1.0. The deep losses displayed in Figure 5.11 result from purchases and injections made under the incorrect assumption that any injected natural gas could be entirely withdrawn and sold at some later stage.

This is a basic mismatch between operational and trading decisions in the presence of capacity limits. As a consequence of this mismatch, more injection capacity is not necessarily beneficial for DOTP for a given level of withdrawal capacity, when the latter capacity is “low.”
For example, this can occur for WCSFs between 0.1 and 0.4. In other words, when the operations manager and the trader do not coordinate their decisions, it can be better to have a tighter injection capacity when the withdrawal capacity is “too low” because limited injection capacity avoids excessive inventory build up. In contrast, a higher withdrawal capacity appears to be always beneficial for DOTP, since it helps to dispose of previously accumulated inventory.

5.7 Conclusions

This chapter deals with the modeling of commodity storage assets. The analysis is based on an MDP formulation of the classical warehouse problem that encompasses both the fast and the slow storage asset cases. This analysis focuses on (i) the structure of an optimal
policy to this MDP, (ii) its computation, and (iii) the managerial relevance of coordinating inventory trading and operational decisions. The findings from this analysis are as follows: (i) an optimal policy has a basestock level structure in the fast asset case and a basestock target structure in the slow asset case; under some assumptions on the spot price dynamics, the basestock levels/targets decrease in the spot price in each given stage; (ii) when the spot price dynamics follow a discrete space process, the basestock levels in the fast case can be easily computed, while the computation of the basestock targets in the slow case requires an additional lot size assumption on the operational parameters of the asset; (iii) coordinating operational and trading decisions when managing a slow asset is of substantial managerial importance.

5.8 Notes

This chapter is based in part on Charnes et al. [54] for the fast asset case and largely on Secomandi [190] for the slow asset case. These papers include proofs for the results stated here.


Dreyfus [80] shows that the optimal policy of the model of Bellman [10] has a structure analogous to that discussed in §5.3.1. Charnes et al. [54] generalize the analysis of Dreyfus [80] to the case of stochastic prices. Rempala [177] establishes a basestock structure for the case of limited injection capacity. Secomandi [190] shows the optimality of the basestock target structure in the general case when both the injection and the withdrawal capacities are slow. Secomandi [191] provides an opportunity cost view of basestock optimality.

The basestock structure is related to the basestock results available in the inventory management literature (Zipkin [234], Porteus [173]),
whose focus is on managing inventory to provide adequate service levels to customers in the face of demand uncertainty, rather than optimizing the inventory trading decisions of merchants that operate in wholesale commodity markets to take advantage of price fluctuations.

The analysis of the price monotonicity of the optimal basestock targets and their computation in §§5.4-5.5 is from Secomandi [190]. Secomandi et al. [194] show that it is possible to generalize Proposition 4 by dispensing of Assumption 3. That is, Secomandi et al. [194] establish that so long as the capacity limits and the minimum and maximum inventory levels are integer multiples of the lot size $Q$, then the value function is piecewise linear concave in inventory with break points that are integer multiple of $Q$ even if Assumption 3 is not satisfied. It then follows that the basestock targets can be taken to be integer multiples of $Q$ even without making this assumption. That is, Assumption 3 is superfluous in Proposition 4. However, this assumption is critical to make model (5.4)-(5.6) a discrete space SDP, which can be solved by backward dynamic programming.

Kobasa [140, pp. 8-9], Tek [207, Chapter 3], Maragos [158, p. 435], and Geman [101, pp. 304-307] discuss the capacitated nature of natural gas storage assets. Natural gas storage contracts can include “ratcheted” injection or withdrawal capacity specifications. That is, the injection or withdrawal capacities are step functions of inventory. This phenomenon is due to the increased difficulty of injecting (withdrawing) natural gas when the storage facility becomes fuller (emptier). Parsons [170] points out that in this case the optimal inventory trading policy can have a more complicated structure than the basestock structure discussed in §5.3.

The analysis of the managerial importance of coordinating operational and trading decisions is from Secomandi [190]. In particular, the estimates of the parameters of the mean-reverting model reported in Table 5.2 and used in this analysis are from Wang [221]. Other authors who use a mean-reverting spot price model for natural gas storage valuation include de Jong and Walet [69], Boogert and de Jong [25], and Manoliu [156]. Chen and Forsyth [58], Kaminski et al. [131], Thompson et al. [212], and Carmona and Ludkovski [46] value natural gas storage using continuous-time stochastic control methods.
Zhou et al. [232] study the storage of electricity and generalize the work of Charnes et al. [54] by allowing the price of electricity to be negative. Schneider [184] models power spot prices that can be negative. Jafarizadeh [124] studies oil storage.

The notation used in this chapter is an adaptation of the notation used in Secomandi [190] and Lai et al. [146]. For consistency, the same notation is used in Chapter 6, which is based on Lai et al. [146]. The drawback of this consistency is that the notation of this monograph is not entirely consistent with both the notation of Secomandi [190] and the notation of Lai et al. [146].
This chapter investigates how commodity storage assets are managed in practice, focusing on natural gas storage. Section 6.1 motivates this study and summarizes the main results of this chapter. Section 6.2 introduces the natural gas storage valuation problem and formulates it as an exact SDP. In contrast to the storage model considered in Chapter 5, this model includes in its states the entire futures curve rather than only the spot price. This difference makes the computation of an optimal storage policy intractable. Section 6.3 introduces heuristic policies commonly used in practice. Section 6.4 presents a tractable approximate dynamic programming (ADP) model that yields yet another heuristic policy, and discusses the computations of two upper bounds on the value of an optimal policy. Section 6.6 benchmarks the performance of these heuristics policies and upper bounds on a set of realistic instances. Section 6.7 concludes. Section 6.8 gives pointers to the literature.
6.1 Introduction

In the United States, current federal regulation separates ownership of natural gas storage facilities from the control of their capacity. That is, the owners of this capacity must make it available to users in an open access fashion via storage contracts. This chapter deals with the benchmarking of methods used to value and manage these contracts in practice. That is, in this chapter the storage asset is a rental contract for the capacity of a natural gas storage facility.

Natural gas merchants manage such contracts as real options on natural gas prices, whose values derive from the intertemporal trading of natural gas allowed by storage. Organized markets such as NYMEX and ICE trade natural gas related financial instruments, including futures and options on futures (see §2.2 in Chapter 2). Practitioners can use them to price natural gas storage contracts using risk-neutral valuation techniques (see Chapter 3).

As discussed in Chapter 5, valuing a storage asset, a contract here, entails dynamic optimization of inventory trading decisions with capacity constraints in the face of uncertain natural gas price dynamics. Stochastic dynamic programming is the natural approach to this valuation problem. However, this approach is tractable only when a low-dimensional spot price model is employed to describe the stochastic evolution of the price of natural gas (see §5.5 in Chapter 5).

Natural gas storage traders do not appear to like this low dimensional price modeling approach, due to doubts about whether the futures and option price dynamics implied by such price models are consistent with the dynamics of the NYMEX or ICE futures and options they trade to hedge price risk. For example, in discussing the viability of using stochastic dynamic programming for natural gas storage valuation, Eydeland and Wolyniec [86, p. 367] make the following observations:

Great care must be taken when specifying and calibrating spot processes for the use in optimization, so that they are consistent with the hedging strategy to be pursued. Additionally, even for a given set of forward in-
formation, the critical surface may exhibit unstable behavior that renders it of limited use as a hedging tool.

According to the practice-based literature (Maragos [158, pp. 449-453], Eydeland and Wolyniec [86, pp. 351-367], Gray and Khandelwal [112, 111]), the preferred modeling approach for many natural gas storage traders is to model the full dynamics of the futures term structure using high-dimensional futures price evolution models, such as some of those discussed in §4.3 in Chapter 4. Unfortunately, this modeling choice makes stochastic dynamic programming computationally intractable and necessitates the use of alternative valuation approaches based on suboptimal but computationally tractable operating policies. According again to this practice-based literature, practitioners typically value natural gas storage based on such heuristic operating policies. Interestingly, commercial software products for the valuation of natural gas storage (see, e.g., FEA [89], KYOS [145]) include heuristic valuation models that can accommodate high-dimensional futures price evolution models.

One such heuristic combines linear programming and spread option valuation methods with or without periodic reoptimization embedded in Monte Carlo simulation of the futures curve. We label the two resulting heuristic policies LP and RLP, where LP abbreviates linear program and the prefix R indicates reoptimization. Another heuristic is based on reoptimization of a deterministic dynamic program, which computes the intrinsic value of storage, within Monte Carlo simulation of the futures curve. We label this heuristic policy RI, where I abbreviates intrinsic and R again stands for reoptimization.

This chapter has two goals: (i) investigate the effectiveness of these heuristics, and (ii) discuss how to improve on their performance when possible. Achieving these goals requires the availability of a good upper bound on the value of storage and the development of alternative heuristics.

An upper bound on the value of storage is obtained by using as a starting point the value function of a tractable approximate dynamic programming (ADP) method to value a commodity storage asset given a high-dimensional model of the evolution of the futures curve. This
Table 6.1 Models and Policies Studied in this Paper.

<table>
<thead>
<tr>
<th>Model</th>
<th>Without Reoptimization</th>
<th>With Reoptimization</th>
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<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>RI</td>
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<tr>
<td>LP</td>
<td>LP</td>
<td>RLP</td>
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<tr>
<td>ADP</td>
<td>ADP</td>
<td>RADP</td>
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</tbody>
</table>

The optimal policy of the ADP model is a heuristic policy for the exact SDP. The value function of the ADP model is an approximation of the value function of the exact SDP. An upper bound is computed by applying the information relaxation and duality approach developed by Brown et al. [39]. Denote this upper bound by DUB, where D and UB abbreviate dual and upper bound, respectively. A simpler approach is to compute a perfect information upper bound, labeled PIUB. PIUB serves as a benchmark for DUB. An additional lower bound is obtained by sequentially reoptimizing the ADP model within Monte Carlo simulation of the futures curve to obtain the RADP policy. Thus, one obtains two ADP-based lower bounds, the ADP and the RADP lower bounds. Table 6.1 summarizes the models and policies analyzed in this chapter.

On a set of realistic instances based on NYMEX price data and additional data available in the energy trading literature, DUB is found to be much tighter than PIUB. DUB is thus used to benchmark on these

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1 This structure extends the basestock structure discussed in §5.3 in Chapter 5. As pointed out in §6.8, an optimal policy for the full-dimensional storage asset valuation SDP is characterized by an extended version of this structure.
instances the practice-based heuristics and the ADP and RADP lower bounds. The intrinsic value of storage accounts for a relatively small amount of the total value of storage, but its computation is extremely fast. The LP heuristic is also very fast and captures significantly more value than the intrinsic value of storage, but its suboptimality is large when compared to DUB. The ADP lower bound exhibits less suboptimality, in most cases, but higher computational requirement than these heuristics.

Reoptimization improves the performance of all the considered policies. In almost all the instances, the RLP, RI, and RADP lower bounds are all nearly optimal when compared to DUB. Thus, DUB is a fairly tight upper bound and plays a critical role in benchmarking the various heuristics. Of course, all these reoptimized lower bounds are more expensive to compute. In particular, the computational requirement of the RADP lower bound is higher than those of the other lower bounds. Overall, the RI heuristic strikes a good compromise between computational efficiency and valuation quality on our instances. However, in some cases the RADP policy can improve on the performance of the RI policy at the expense of increased computational effort.

The results discussed in this chapter are useful for natural gas storage traders because they provide scientific validation, support, and guidance for the use of heuristic storage valuation models in practice. Moreover, these results remain substantially similar, provided that reoptimization is used, when the seasonality in the NYMEX natural gas futures curves used to obtain them is (artificially) eliminated. This finding suggests that the insight into the benefit of reoptimization has potential relevance for the merchant storage of other commodities, such as metals, oil, and petroleum products (see §2.1 in Chapter 2), whose futures curves do not exhibit the pronounced seasonality of the natural gas futures curve.

6.2 Valuation Problem and Exact SDP

This section describes the natural gas storage valuation problem and formulates an exact SDP of this problem that extends the SDP (5.4)-
A natural gas storage contract gives a merchant the right to inject, store, and withdraw natural gas at a storage facility up to given limits during a finite time horizon. The injection and withdrawal capacity limits are expressed in mmBtus per unit of time, e.g., day or month. There are also limits on the minimum and maximum amounts of the natural gas inventory that the merchant can hold under such a contract. There are proportional charges and fuel losses associated with injections and withdrawals.

The wholesale natural gas market in North America features about one hundred geographically dispersed markets for the commodity. NYMEX and ICE trade financial contracts associated with about forty of these markets (see §2.2 and §2.4 in Chapter 2). The most liquid market is Henry Hub, Louisiana, which is the delivery location of the NYMEX natural gas futures contract. NYMEX also trades options on this contract. Moreover, NYMEX and ICE trade basis swaps, which are financially settled forward locational price differences relative to Henry Hub. Thus, these financial instruments make practical the risk-neutral valuation of natural-gas related cash flows.

The quantity of interest is the value of a given natural gas storage contract at the time of its inception. This value depends on how the natural gas price changes over time because a merchant uses storage to support trading in the natural gas market: buying natural gas and injecting it into the storage facility at a given point in time, storing it for some time, and withdrawing it out of the facility and selling it at a later point in time. A storage contract can be valued as the discounted risk-neutral expected value of the cash flows from optimally operating it during its tenor, given its operational constraints (see Chapters 3 and 5).

Of primary interest to traders is the value of the “forward” or “monthly-volatility” component of a storage contract (Maragos [158, p. 440], Eydeland and Wolyniec [86, p. 365]). The value of this component can be hedged by trading futures contracts, and corresponds to

\[(5.6)\text{ considered in }\S 5.2\text{ in Chapter }5.\]
the value of the cash flows associated with making natural gas trading decisions on a monthly block basis. Thus, attention is restricted to the valuation of monthly cash flows.

The storage contract tenor spans $N$ futures maturities in set $\mathcal{I} \equiv \{0, \ldots, N-1\}$. Inventory trading decisions are made at each maturity time $T_i$ with $i \in \mathcal{I}$. The notation $F_{T_i,T_j}$ denotes the futures price at time $T_i$ with maturity at time $T_j$, $\forall i, j \in \mathcal{I}$, $j \geq i$; $F_{T_i,T_i}$ is the spot price at time $T_i$. With some abuse of notation, for the most part the notation $F_{T_i,T_j}$ is replaced with the alternative notation $F_{i,j}$ to simplify the exposition. The former notation is useful when dealing with continuous-time dynamics of futures prices, the latter simplifies the writing of discrete-time dynamic programs. Define the futures curve at time $T_i$ with maturities through time $T_N$ as $F_i := (F_{i,j}, j \in \mathcal{I}, j \geq i), \forall i \in \mathcal{I}$; by convention $F_N := 0$. Notice that $F_i$ includes the spot price at time $T_i$. Define the futures curve at time $T_i$ without including the spot price as $F_i' := (F_{i,j}, j \in \mathcal{I}, j > i), \forall i \in \mathcal{I} \setminus \{N-1\}$; $F_{N-1}' := 0$.

The numerical study in this chapter is based on a string model of futures price evolution (see §4.3 in Chapter 4). This approach is representative of the high-dimensional futures curve evolution models discussed in the practice-based literature. The risk-neutral dynamics of the natural gas futures price associated with each maturity date $T_i$ is described by a driftless Geometric Brownian Motion, with maturity-specific constant volatility $\sigma_i > 0$ and Standard Brownian Motion increment $dZ_{i,t}$. Moreover, the Standard Brownian Motion increments corresponding to two different maturity dates $T_i$ and $T_j$ are instantaneously correlated with constant correlation coefficient $\rho_{i,j} \in (-1, 1)$, and $\rho_{i,i} := 1$. This model is

$$\frac{dF_{t,T_i}}{F_{t,T_i}} = \sigma_i dZ_{i,t}, \forall i \in \mathcal{I}, \quad (6.1)$$

$$dZ_{i,t}dZ_{j,t} = \rho_{i,j} dt, \forall i, j \in \mathcal{I}, i \neq j. \quad (6.2)$$

The storage valuation problem is formulated as an SDP by modifying the one-factor spot-price SDP (5.4)-(5.6) presented in Chapter 5. The inventory review period is monthly, so that each review time corresponds to a futures price maturity. The states in stage $i$ of SDP (5.4)-(5.6) is modified by replacing the spot price $s_i$ with the futures
curve $F_i$, so that the value function in stage $i$ is written as $V_i(x, F_i)$. According to the practice-based literature, natural gas storage contracts typically do not seem to include a holding cost. Thus, the unit holding cost, $h$, is set to zero. For simplicity, let the one-stage risk-free discount factor be constant across stages, and denote it by $\delta$.

The minimum inventory level is normalized to $0$, $x := 0$. Thus, the set of feasible inventory levels $\mathcal{X}$ is $[0, \infty]$. The feasible injection action set at each inventory level is identical to the one defined in Chapter 5. In contrast, since the minimum inventory level is zero, the feasible withdrawal action set at inventory level $x$, $A_W(x)$, is $[C_W \vee (-x), 0]$. The feasible action set is updated accordingly.

The modified SDP for storage valuation is

\[
V_N(x_N, F_N) := 0, \forall x_N \in \mathcal{X}, \quad (6.3)
\]
\[
V_i(x_i, F_i) = \max_{a \in A(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN} \left[ V_{i+1}(x_i + a, \tilde{F}_{i+1}) | F_i \right], \quad \forall i \in \mathcal{I}, (x_i, F_i) \in \mathcal{X} \times \mathbb{R}^{N-i}, \quad (6.4)
\]

where as in Chapter 5 the function $r(a, s_i)$ is the immediate payoff of action $a$ given the spot price $s_i$, and where $\mathbb{E}^{RN}$ in (6.4) denotes expectation taken with respect to the risk-neutral distribution of random vector $\tilde{F}_{i+1}$ conditional on $F_i$, which is sufficient to evaluate this expectation under model (6.1)-(6.2). In the remainder of this chapter, and consistent with the notation used in Chapter 5, a tilde on top of a symbol denotes a random entity.

In practice, the number of monthly maturities $N$ associated with natural gas storage contracts at initiation is at least twelve, so that model (6.3)-(6.4) is in general computationally intractable because of its high-dimensional state space.

### 6.3 Practice-based Heuristics

This section describes the practice-based policies $LP$ and $I$, and their reoptimization versions $RLP$ and $RI$. 
6.3.1 Model Based on Spread Options

The LP policy is based on spread option valuation and linear programming. A spread option is an option on the difference between two prices with a positive strike price (see §2.4 in Chapter 2). The LP policy uses spread options on the difference between the futures price $F_{i,j}$ and the spot price $F_{i,i}$ on a future date $T_i$, with $i < j$, adjusted for the time value of money and fuel consumption, and a strike price equal to the sum of the time $T_i$ injection marginal cost and the time $T_j$ withdrawal marginal cost discounted back to time $T_i$. Such an option is referred to here as the $i$-$j$ spread option. Its time $T_0$ value is

$$
\delta^i E_{RN} \left\{ \left[ \delta^{j-i} \phi^W F_{i,j} - \phi^F F_{i,i} - (\delta^{j-i} e^W + c_i) \right]^+ \right| F_{0,i}, F_{0,j} \right\}, \quad (6.5)
$$

where $\{\cdot\}^+ := \max\{\cdot, 0\}$, and is denoted by $S^{i,j}_0(F_{0,i}, F_{0,j})$. This value is the time $T_0$ value of injecting one unit of natural gas at time $T_i$ and withdrawing it at time $T_j$ provided that the value of this trade is nonnegative at time $T_i$.

The LP policy works with portfolios of spread options $\{q_{i,j}, i, j \in I, i < j \}$ where $q_{i,j}$ is the notional amount of natural gas associated with spread option $i$-$j$. More explicitly, such a portfolio includes notional amounts for spread options whose injections and withdrawals are associated with maturities $0, 1, \ldots, N - 2$, and $1, 2, \ldots, N - 1$, respectively. The LP policy also includes a spot sale $y_0$ of some or all of the inventory available at time $T_0$.

The initial step of the LP policy is to approximate the value of storage at time $T_0$ by constructing a portfolio of spread options and a spot sale as an optimal solution to the LP (6.6)-(6.14) below. The decision variables in this LP are the notional amounts in set $\{q_{i,j}, i, j \in I, i < j \}$; the inventory levels in set $\{x_i, i \in I \setminus \{0\} \cup \{N\}\}$; and the spot sale $y_0$. This LP, which only depends on the time $T_0$ information set $\{x_0, F_0\}$, is

$$
U_0^{LP}(x_0, F_0) := \max_{y_0, q, x} s_0 y_0 + \sum_{i \in I} \sum_{j \in I, i < j} S^{i,j}_0(F_{0,i}, F_{0,j}) q_{i,j} \quad (6.6)
$$

Typically storage contracts do not entail a positive initial inventory, but this decision variable is useful in the reoptimization version of this LP.
6.3. Practice-based Heuristics

\[ x_{i+1} = x_i + \sum_{j \in I, j > i} q_{i,j} - y_0 1\{i = 0\} \]
\[ - \sum_{j \in I, j < i} q_{j,i}, \forall i \in \mathcal{I}, \quad (6.7) \]
\[ x_i \leq \overline{x}, \forall i \in \mathcal{I} \setminus \{0\} \cup \{N\}, \quad (6.8) \]
\[ \sum_{j \in I, j > i} q_{i,j} \leq C^I, \forall i \in \mathcal{I} \setminus \{N-1\}, \quad (6.9) \]
\[ y_0 \leq -C^W, \quad (6.10) \]
\[ \sum_{i \in I, i < j} q_{i,j} \leq -C^W, \forall j \in \mathcal{I} \setminus \{0\}, \quad (6.11) \]
\[ y_0 \geq 0, \quad (6.12) \]
\[ q_{i,j} \geq 0, \forall i, j \in \mathcal{I}, i < j, \quad (6.13) \]
\[ x_i \geq 0, \forall i \in \mathcal{I} \setminus \{0\} \cup \{N\}. \quad (6.14) \]

The objective function (6.6) is the value of the portfolio that includes the spot sale and the portfolio of spread options. Constraint sets (6.7) and (6.8) express inventory balance and bounding conditions, respectively (\(1\{\cdot\}\) in (6.7) is the indicator function, which is equal to 1 if its argument is true and 0 otherwise). Constraint sets (6.9)-(6.11) enforce capacity constraints. Constraint sets (6.12)-(6.14) impose nonnegativity conditions on the decision variables. The quantity \(U_{LP}^0(x_0, F_0)\) is the maximized portfolio value.

Model (6.6)-(6.14) requires prices for the spread options that appear in the objective function (6.6). Once these prices are known, this model can be optimally solved very efficiently. These prices can be obtained in two ways: from (i) market quotes or (ii) a model. Market prices are not available initially if a liquid market for these spread options is not active. In addition, spread option prices are not available for future stages when using the reoptimized version of model (6.6)-(6.14) within a Monte Carlo simulation of the futures curve (see §6.3.3). Hence, in these cases spread option prices must be obtained from a model. When using the string model (6.1)-(6.2), there is no closed-form spread option pricing formula. However, spread option prices can be numerically computed or, alternatively, as in the analysis of §6.6, they can be approximated using closed-form formulas, such as Kirk’s formula (see
Let \( y_0^{LP}(x_0, F_0) \) and \( \{q_{i,j}^{LP}(x_0, F_0), i, j \in \mathcal{I}, i < j \} \) be a portfolio that optimally solves the LP model (6.6)-(6.14). This portfolio can be used to construct a feasible policy for model (6.3)-(6.4). This is the LP policy.

To describe the LP policy, define the following quantities:

\[
q_{i,j}^{LP,+}(F_{i,i}, F_{i,j}) := \begin{cases} 0, & \text{if } \delta^{j-i} F_{i,j} - \left( \phi^i F_{i,i} + \delta^{j-i} c^W + c^I \right) \leq 0, \\ -q_{i,j}^{LP}(x_0, F_0), & \text{otherwise}. \end{cases}
\]

These quantities depend on \( \{x_0, F_0\} \), but this dependence is suppressed from the notation for ease of exposition. Given the sequence of futures curves observed up to and including time \( T_i \), the LP policy uses the quantities defined by (6.15) and the optimal spot-sale \( y_0^{LP}(x_0, F_0) \) to obtain the following feasible inventory change initiated at time \( T_i \):

\[
-y_0^{LP}(x_0, F_0)1\{i = 0\} - \sum_{j \in \mathcal{I}, i < j} q_{j,i}^{LP,+}(F_{j,j}, F_{j,i}) + \sum_{j \in \mathcal{I}, i > j} q_{i,j}^{LP,+}(F_{i,i}, F_{i,j}).
\]

This decision does not depend on the inventory level \( x_i \). Nevertheless the LP policy is feasible for model (6.3)-(6.4), as now discussed.

Denote the time \( T_0 \) value of the LP policy by \( V_0^{LP}(x_0, F_0) \), which can be estimated using Monte Carlo simulation of the futures curve evolution. Proposition 5 states that the optimal portfolio value \( U_0^{LP}(x_0, F_0) \) is no greater than the value of the LP policy \( V_0^{LP}(x_0, F_0) \), and that both of these values are lower bounds on the value of storage \( V_0(x_0, F_0) \).

\[
\text{Proposition 5 (LP policy value).} \quad \text{It holds that } U_0^{LP}(x_0, F_0) \leq V_0^{LP}(x_0, F_0) \leq V_0(x_0, F_0).
\]

The first inequality in Proposition 5 follows from observing that the optimal portfolio value \( U_0^{LP}(x_0, F_0) \) underestimates the value of the LP policy \( V_0^{LP}(x_0, F_0) \), because the objective function (6.6) “double counts” the costs and fuel losses of spread options with overlapping injections and withdrawals that the LP policy combines into a single
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inventory adjustment. The basic idea behind the second inequality is showing that constraints (6.7)-(6.14) are sufficient for the feasibility of the LP policy for the SDP (6.3)-(6.4).

6.3.2 Model Based on Forward Contracts

Another model of interest is the so called intrinsic value model, which computes the value of storage that can be attributed to seasonality, as expressed by the futures curve in the initial stage. This model is the following deterministic dynamic program:

\[ U_N(x_N; F_0) := 0, \forall x_N \in \mathcal{X}, \]

\[ U_i(x_i; F_0) = \max_{a \in A(x_i)} r(a, F_0, i) + \delta U_{i+1}(x_i + a; F_0), \forall i \in \mathcal{I}, \forall x_i \in \mathcal{X}. \]

This model computes an optimal policy that only considers the price information available at time \( T_0 \). This is the I policy, which corresponds to a sequence of purchases-and-injections or withdrawals-and-sales, one for each stage, determined based only on date \( T_0 \) information. The cash flows associated with this policy can be secured at time \( T_0 \) by transacting in the forward market for natural gas at this time. The time \( T_0 \) value of the I policy is \( U_0(x_0; F_0) \).

6.3.3 Models Based on Reoptimization

It is typically possible to improve the performance of the LP and I policies by reoptimizing their associated linear and dynamic programs at each maturity to take advantage of new price and inventory information that becomes available over time, implementing the action pertaining to the maturity when the reoptimization is performed, and repeating this process up to and including the last maturity. For brevity, the details of this process are not discussed here. The time \( T_0 \) values of the reoptimization versions of the LP and I policies, that is, the RLP and RI policies, are clearly lower bounds on the value of storage and can be estimated by Monte Carlo simulation of the futures curve.
6.4 ADP Model

This section discusses the ADP model, some structural results for this model, and how it can be used to compute lower and upper bounds on the value of storage.

6.4.1 ADP Policy

To reduce the computationally intractable and exact SDP (6.3)-(6.4) to a computationally tractable and approximate model, the ADP approach discussed below uses information and value function approximations, which reduce the high dimensionality of the SDP (6.3)-(6.4) in order to compute an approximate and feasible policy for this model.

The ADP model is introduced by reformulating the exact SDP. To this aim, define the futures curve at time $T_i$ excluding both the spot and prompt month futures prices as $F_i^{''}$ := $(F_{i,j}, j \in \mathcal{I}, j > i + 1)$, $\forall i \in \mathcal{I} \setminus \{N - 2, N - 1\}$; $F_N^{''}$ := 0. Also define as follows the expected value function in stage $i$ and state $(x_i, s_i, F_i^{''})$ given that the stage $i$ inventory level $x_i$ and spot price $s_i$ and the stage $i - 1$ futures curve $F_{i-1}^{''}$ are known but the stage $i$ futures curve $F_i'$ is unknown:

$$V_i'(x_i, s_i, F_i') := \mathbb{E}^{RN} \left[ V_i(x_i, s_i, \tilde{F}_i') | s_i, F_i^{''} \right],$$

for all $i \in \mathcal{I} \setminus \{0\}$ and $(x_i, s_i, F_i^{''}) \in \mathcal{X} \times \mathbb{R}_+^{N-i}$. Thus, the recursion (6.4) in stage $i \in \mathcal{I} \setminus \{N - 1\}$ and state $(x_i, F_i)$ can be equivalently expressed as

$$V_i(x_i, F_i) = \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN} \left[ V_{i+1}(x_i + a, \tilde{F}_{i+1}) | F_i' \right],$$

$$= \max_{a \in \mathcal{A}(x_i)} r(a, s_i)$$

$$+ \delta \mathbb{E}^{RN} \left[ \mathbb{E}^{RN} \left[ V_{i+1}(x_i + a, s_{i+1}, \tilde{F}_{i+1}) | s_{i+1} = s_{i+1}, F_i' \right] | F_i, i+1 \right],$$

$$= \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN} \left[ V_{i+1}(x_i + a, s_{i+1}, F_i') | F_i, i+1 \right],$$

where the second equality follows from iterated expectation, that is, breaking the one-step conditioning on $F_i'$ into an equivalent two-step
conditioning on $F_{i,i+1}$ and then $F''_i$. The maximization in (6.20) is computationally intractable in part because of the dimensionality of the futures curve $F''_i$, but this dimensionality can be reduced based on the following approximations:

1) **Information.** Replace $F''_i$ in (6.20) with $(F_0,i+2, \ldots, F_{N-1},i)$, which is information known at time $T_0$. This effectively reduces the dimensionality of maximization (6.20) because the third argument of the function $V_{i+1}(\cdot,\cdot,\cdot)$, $(F_0,i+2, \ldots, F_{N-1},i)$, is known at time $T_0$.

2) **Value function.** Replace the unknown function $V_{i+1}(\cdot,\cdot,(F_0,i+2, \ldots, F_{N-1},i))$ with an approximation. There are many functions that could be used to approximate this unknown function. The particular function used here, $U^{ADP}_{i+1}(\cdot,\cdot)$, is recursively defined as now described.

At the final stage, the boundary conditions are

$$U^{ADP}_N(x_N,s_N) := 0, \forall x_N \in \mathcal{X}. \quad (6.21)$$

At earlier stages, introduce the approximate value function

$$u^{ADP}_i(x_i,s_i,F_{i,i+1}) := \max_{a \in \mathcal{A}(x_i)} \{ r(a,s_i)$$

$$+ \delta \mathbb{E}^{RN} \left[ U^{ADP}_{i+1}(x_{i} + a, s_{i+1}) \mid F_{i,i+1} \right] \}, \quad (6.22)$$

for all $i \in I$ and $(x_i,s_i,F_{i,i+1}) \in \mathcal{X} \times \mathbb{R}_2^+$, and, mimicking (6.19), define

$$U^{ADP}_i(x_i,s_i) := \mathbb{E}^{RN} \left[ u^{ADP}_i(x_i,s_i,F_{i,i+1}) \mid s_i, F_{0,i+1} \right], \quad (6.23)$$

for all $i \in I$ and $(x_i,s_i) \in \mathcal{X} \times \mathbb{R}_+$, with $F_{0,N} = F_{N-1,N} := 0$.

Expressions (6.21)-(6.23) define the ADP model, which can be used to generate a feasible policy for the exact SDP (6.3)-(6.4) through the maximization on the right-hand side of (6.22). Call the resulting heuristic policy the ADP policy. Denote by $a^{ADP}_i(x_i,s_i,F_{i,i+1})$ the action

---

4 Under the string model (6.1)-(6.2), the futures price $F_{i,i+1}$ and the futures curve $F''_i$ are sufficient to evaluate the outer and inner expectations, respectively.
taken by this policy in stage \( i \) and state \((x_i, s_i, F_{i,i+1})\); if the set

\[
\arg \max_{a \in \mathcal{A}(x_i)} r(a, s_i) + \delta \mathbb{E}^{RN}[U_{i+1}^{ADP}(x_i + a, \tilde{s}_{i+1}) \mid F_{i,i+1}],
\]

(6.24)
is not a singleton, the action \( a_i^{ADP}(x_i, s_i, F_{i,i+1}) \) is set equal to the element in this set with the smallest absolute value.

Making this ADP approach computationally tractable requires (i) using finite sets for the possible values of the spot price \( s_i \) and the futures price \( F_{i,i+1} \); (ii) suitably discretizing the dynamics of this futures price into the next stage spot price; (iii) interpolating to obtain values of \( U_{i+1}^{ADP}(x_i + a, \cdot) \) that are otherwise unavailable, given the discretization in step (i), when computing the expectation in (6.24) using the discretized evolution of the futures price \( F_{i,i+1} \) into the spot price \( s_{i+1} \) in step (ii); and (iv) mapping the pair \((s_i, F_{i,i+1}) \in \mathbb{R}^2_+\) to one of the corresponding pairs in step (i), when implementing the policy obtained by solving (6.24) in this manner (see Lai et al. [146] for details).

The value \( U_0^{ADP}(x_0, F_0) \) computed by the ADP model (6.22)-(6.23) is not typically equal to the value of the ADP policy evaluated under the full information available in the SDP (6.3)-(6.4), which is denoted as \( V_0^{ADP}(x_0, F_0) \). When valuing the ADP policy one has access to all the relevant price information (that is, the entire futures curve at a given time), but when computing the ADP policy only partial information is used (that is, the current spot and prompt month futures prices at a given time and the futures curve in the initial stage). The value of the ADP policy can be estimated by Monte Carlo simulation of the futures curve.

The following result holds because the ADP policy is feasible for the exact model.

**Proposition 6 (ADP policy value).** It holds that \( V_0^{ADP}(x_0, F_0) \leq V_0(x_0, F_0) \).

The computation the ADP policy can benefit from exploiting its stage- and price-state dependent basestock target structure, which is analogous to the structure established in Theorem 5.2 in Chapter 5. That is, in each stage there exist two basestock targets, which depend
on the available price information, that is, the pair \((s_i, F_{i,i+1})\), such that it is optimal to buy-and-inject (respectively, withdraw-and-sell) to get as close as possible to the lower (respectively, higher) target from any inventory level below (respectively, above) such target; doing nothing is optimal at inventory levels in between these targets.

Moreover, the basestock targets decrease in the spot price \(s_i\) for each given stage \(i\) and futures price \(F_{i,i+1}\). Further, suppose that the distribution of the spot price in the next stage conditional on the prompt month futures price in the current stage, denoted by \(\Phi(s_{i+1}|F_{i,i+1})\), stochastically increases in the latter quantity; that is, this distribution satisfies the property that \(1-\Phi(s_{i+1}|F_{i,i+1})\) increases in \(F_{i,i+1}\) for each given \(s_{i+1}\). For example, this assumption is satisfied by the multidimensional Black model (6.1)-(6.2). Then, the basestock targets increase in the futures price \(F_{i,i+1}\) for each given stage \(i\) and spot price \(s_i\). These properties can be interpreted as follows: as the spot price increases it is optimal to purchase less and sell more; as the prompt month futures price increases it is optimal to purchase more and sell less. The first of these complementarity relationships (Topkis [215, pp. 92-93]) is consistent with the monotonicity result stated in Theorem 5.3 in Chapter 5.

### 6.4.2 RADP Policy

The ADP policy is computed once at time \(T_0\). The reoptimization version of this policy (the RADP policy) involves re-solving the ADP model in each stage after the initial stage given the information available at that time. In other words, model ADP is reoptimized at each time \(T_j, j \in \mathcal{I} \setminus \{0\}\), by using the futures curve \(F_j\) in place of \(F_0\). Specifically, \(F_{0,i+1}\) is replaced with \(F_{j,i+1}\) in (6.23). The time \(T_0\) value of the resulting RADP policy is typically different, and in fact higher, than that of the ADP policy. In other words, the RADP policy can typically benefit from sequential reoptimization. Since the RADP policy is feasible for the exact SDP, its time \(T_0\) value is a lower bound on the storage contract value, \(V_0(x_0, F_0)\).
6.5 Upper Bounds

The value function of the ADP model can be used to compute an upper bound on the value of storage. Define the penalty terms

$$p_i^{ADP}(x_i, a, s_{i+1}, F_{i,i+1}) := U_{i+1}^{ADP}(x_i + a, s_{i+1}) - \mathbb{E}^{RN}[U_{i+1}^{ADP}(x_i + a, \tilde{s}_{i+1}) | F_{i,i+1}],$$

for all $i \in I$ and $(x_i, a, s_{i+1}, F_{i,i+1}) \in \mathcal{X} \times \mathcal{A}(x_i) \times \mathbb{R}^2_+$, which are based on the value function of the ADP model solved at time $T_0$. These penalty terms have an appealing interpretation in terms of additional value of perfect information. Consider performing action $a$ in stage $i$ given the inventory level $x_i$ and the prompt month futures price $F_{i,i+1}$. If the next stage spot price $s_{i+1}$ is known, then the function $U_{i+1}^{ADP}(x_i + a, s_{i+1})$ approximates the value of the resulting inventory level $x_i + a$ in stage $i + 1$. If this spot price is not known, the approximate value of this inventory level is the risk-neutral expectation $\mathbb{E}^{RN}[U_{i+1}^{ADP}(x_i + a, \tilde{s}_{i+1}) | F_{i,i+1}]$. The additional value of perfect information is thus the difference on the right-hand side of (6.25). These quantities are used to penalize only the availability of hindsight information in the DUB model discussed next; that is, they serve the purpose of “Lagrange” multipliers (duals) only on information one is not supposed to know. In other words, they are dual feasible penalties.

Denote by $P_0$ a sequence of pairs of spot and prompt-month future prices for maturities 0 through $N - 1$; that is, $P_0 := ((s_i, F_{i,i+1}))_{i=0}^{N-1}$. Given $P_0$, solve the DUB model

$$U_N^{DUB}(x_N; P_0) := 0, \quad \forall x_N \in \mathcal{X},$$

$$U_i^{DUB}(x_i; P_0) = \max_{a \in \mathcal{A}(x_i)} r(a, s_i) - p_i^{ADP}(x_i, a, s_{i+1}, F_{i,i+1}) + \delta U_{i+1}^{DUB}(x_i + a; P_0),$$

$$\forall i \in I, \quad x_i \in \mathcal{X}.$$  (6.27)

This is a perfect price information model with immediate rewards penalized according to the penalties (6.25). Define

$$V_0^{DUB}(x_0, F_0) := \mathbb{E}^{RN}[U_0^{DUB}(x_0; \bar{P}_0) | F_0].$$  (6.28)
The quantity $V_0^{DUB}(x_0, F_0)$ is DUB, which is an upper bound on the storage contract value, as stated in Proposition 7, and can be estimated by Monte Carlo simulation of the futures curve.

**Proposition 7 (DUB).** It holds that $V_0(x_0, F_0) \leq V_0^{DUB}(x_0, F_0)$.

Intuitively, DUB is an upper bound on the storage contract value because it is obtained from averaging the value functions in the initial stage and state of perfect information models in which only hindsight information is penalized by penalties that average to zero but affect the result of the optimization on the right-hand side of (6.27). Moreover, if one could use the unknown value function of the exact SDP to define these penalties, which in this case would depend on the futures curves $F_{i+1}$ and $F_i'$ rather than the prices $s_{i+1}$ and $F_{i,i+1}$, the resulting dual upper bound would be tight. Defining these penalties using an approximate, but known, value function, as in (6.25), does not guarantee tightness of the corresponding dual upper bound.

A different upper bound on the storage contract value can be obtained by setting the penalty terms defined in (6.25) equal to zero and proceeding analogously to the computation of upper bound (6.28). This approach yields PIUB, which provides a benchmark for the performance of DUB.

### 6.6 Numerical Results

This section discusses the performance of the models and policies presented in §§6.3-6.4 on a set of realistic benchmark instances.

#### 6.6.1 Instances

The benchmark instances are based on market data and parameter values reported in the energy trading literature.

The two top panels of Figure 6.1 illustrate four futures curves that include the Henry Hub natural gas spot price and NYMEX natural gas futures prices for the first 23 maturities (Henry Hub is the delivery location of the NYMEX natural gas futures contract). These curves were observed on four days, each corresponding to one of the four seasons
of the year: 3/1/2006 (Spring), 6/1/2006 (Summer), 8/31/2006 (Fall), and 12/1/2006 (Winter). The pronounced seasonality in the NYMEX natural gas futures curve is evident in these panels.

The two bottom panels of Figure 5.8 show the Black implied volatilities of the 23 futures prices on each of the four considered trading days obtained from the prices of NYMEX call options on natural gas futures. These panels indicate that futures volatilities “tend” to decrease with longer maturities, which is as expected, but also bring to light what appear to be seasonal volatility patterns that somewhat mirror those displayed by the futures curves.

A historical correlation matrix is estimated using daily futures prices of the first 23 maturities observed between 1/2/1997 and 12/14/2006.

The one-year treasury rates on the four selected dates, as reported
by the U.S. Department of Treasury, are 4.74%, 5.05%, 5.01%, and 4.87%, respectively. They are used as risk-free interest rates to generate the monthly discount factors.

The maximum inventory is normalized to 1 mmBtu. There are three pairs of injection and withdrawal limits, as shown in Table 6.2. The first capacity pair roughly reflects the capacities in Example 8.11 in Eydeland and Wolyniec [86, p. 355]. The other two capacity pairs correspond to multiplying the injection and withdrawal limits of this capacity pair by 2 and 3, respectively, to model “faster” assets. The injection and withdrawal costs are $0.02 and $0.01 per mmBtu, respectively, and the injection and withdrawal fuel coefficients are 1.01 and 0.99, respectively.

The distinguishing features of the benchmark instances are the number of months in the contract tenor (number of stages), the season corresponding to the initial stage (futures curve and volatilities), and the withdrawal/injection limits. The label of an instance encodes this information in the following order:

- Number of stages: 24;
- Season: Sp, Su, Fa, and Wi for Spring, Summer, Fall, and Winter, respectively;
- Capacity pair number: 1, 2, or 3.

In total, there are twelve instances, labeled 24-Sp-1 through 24-Wi-3.

### 6.6.2 Results

The results discussed below are based on evaluating all the policies and the two upper bounds by starting with no initial inventory and by simulating 10,000 futures curve sample paths. Lai et al. [146] provide

<table>
<thead>
<tr>
<th>Number</th>
<th>Injection (mmBtu/month)</th>
<th>Withdrawal (mmBtu/month)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15</td>
<td>0.30</td>
</tr>
<tr>
<td>2</td>
<td>0.30</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>0.45</td>
<td>0.90</td>
</tr>
</tbody>
</table>
a detailed discussion of the computation of these policies and bounds, which is not repeated here for brevity. The PIUB estimates are much weaker than the DUB estimates, ranging from 143% to 258% of the DUB estimates (the average standard errors of the PIUB and the DUB estimates are 1.32% and 0.49% of the DUB estimates, respectively). Thus, in the ensuing discussion, the estimated value of each policy and its standard error are expressed as percentages of the DUB estimates.

**No reoptimization.** Figure 6.2 reports the valuation performance of the ADP, I, and LP policies, that is, the policies without reoptimization. Recall that the I policy computes the intrinsic value of stor-
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age, that is, that part of the storage value that can be attributed to the seasonality in the initial natural gas futures curve, rather than its volatility. Thus, this policy is likely to yield lower valuations than the other policies.

Overall, the ADP and LP policies perform consistently better than the I policy. The performance of the latter policy may be as low as 40% of the DUB value. The ADP policy performs better than the other policies on most of the instances except on the instances 24-Sp-1 and 24-Su-1, where the LP policy outperforms it. The standard errors on the estimates of the values of the policies without reoptimization are typically around 1%. The average standard errors for the ADP, I, and LP policies are 1.18%, 1.16%, and 1.12%, respectively.

To discuss how the valuation performance of a policy depends on the injection/withdrawal capacities, define the range of valuation performances for a policy to be the difference between its minimum and maximum valuation performance figures on each of the three instances that differ only in their injection/withdrawal capacities. For example, the range of the ADP policy on instances 24-Sp-1, 24-Sp-2, and 24-Sp-3 is $(95.12 - 93.11)\% = 2.01\%$. The LP policy is the least sensitive policy with a rough average range of 3%, whereas the ranges of the ADP and I policies are about 4% and 8%, respectively. It appears that the ADP policy is able to capture a larger share of the value of storage for instances with higher injection/withdrawal capacities, while the intrinsic value, relative to the DUB value, becomes smaller when the injection/withdrawal capacities increase (intuitively, increasing these capacities increases the optionality of storage, and, hence, the extrinsic value of storage increases more than its intrinsic value). In particular, the values obtained by the LP policy do not show a monotone pattern as the injection/withdrawal capacities vary.

Table 6.3 reports the statistics on the Cpu times needed to compute and evaluate the different policies without reoptimization.\(^5\)

---

\(^5\)The machine used for these computations is a 64 bits Monarch Emporo 4-Way Tower Server with four AMD Opteron 852 2.6GHz processors, each with eight DDR-400 SDRAM of 2 GB and running Linux Fedora 9. The compiler is g++ version 4.3.0 20080428 (Red Hat 4.3.0-8). All the results are obtained using only one processor. The LPs associated with the LP policy are solved using the Clp linear solver of COIN-OR (www.coin-or.org).
Table 6.3 Statistics on the CPU seconds needed to compute the three policies without reoptimization.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>ADP</th>
<th>I</th>
<th>LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>146.23</td>
<td>0.53</td>
<td>0.64</td>
</tr>
<tr>
<td>Minimum</td>
<td>135.29</td>
<td>0.40</td>
<td>0.46</td>
</tr>
<tr>
<td>Average</td>
<td>140.01</td>
<td>0.46</td>
<td>0.54</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4.54</td>
<td>0.05</td>
<td>0.07</td>
</tr>
</tbody>
</table>

The ADP policy requires on average much more CPU time than the other policies. However, this time can be significantly reduced by using a coarser discretization without significantly affecting the valuation performance of this policy (Lai et al. [146]). The fastest policy to compute and evaluate is the I policy, but the computational requirement of the LP policy is also very small.

**Reoptimization.** Figure 6.3 reports the valuation performance of the RADP, RI, and RLP policies. Reoptimization substantially improves the quality of the ADP, I, and LP policies, yielding nearly optimal policies, which implies that the DUB estimates are fairly tight. In particular, the RADP policy captures at least 96% of the value of storage on all the instances, except on 24-Wi-1 for which this figure is 94.91%. Moreover, the RADP policy performs mostly better than the other policies, with the exception of 24-Sp-2, where it is marginally outperformed by RLP. The performances of the RADP, RI, and RLP policies are all very insensitive to changes in injection/withdrawal capacities with average ranges less than 1%.

Similar to the case without reoptimization, the standard errors of the estimates of the values of the policies with reoptimization are around 1%. The average standard errors for the RADP, RI, and RLP policies are 1.19%, 1.17%, and 1.17%, respectively. It is noteworthy that the gaps between the DUB value and the values of the reoptimization-based policies are somewhat larger on the Winter instances than on the instances associated with the other seasons. This difference is due to DUB being looser on the Winter instances compared to the other
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![Graphs showing spring, summer, fall, and winter capacity pairs.](image)

Fig. 6.3 Valuation performance of the three policies with reoptimization (percent of the DUB estimate).

Table 6.4 Statistics on the Cpu seconds needed to compute the three policies with reoptimization.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>RADP</th>
<th>RI</th>
<th>RLP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum</td>
<td>634.22</td>
<td>24.81</td>
<td>296.12</td>
</tr>
<tr>
<td>Minimum</td>
<td>570.66</td>
<td>23.20</td>
<td>262.88</td>
</tr>
<tr>
<td>Average</td>
<td>595.73</td>
<td>23.90</td>
<td>278.74</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>20.85</td>
<td>0.47</td>
<td>12.26</td>
</tr>
</tbody>
</table>

instances (see the discussion in §6.8).

Table 6.4 reports the statistics on the Cpu times needed to compute the reoptimization-based policies. The RADP policy needs on average...
substantially more CPU time than the other policies, whose computational requirements, however, increase markedly relative to the no-reoptimization case. The policy that can be evaluated the fastest is RI, whose evaluation takes roughly one tenth of the time needed to evaluate the RLP policy, which in turn takes half the time required to evaluate the RADP policy.

Summary. The results discussed in this section bring to light the benefit of combining reoptimization and Monte Carlo simulation of the futures curve for natural gas storage valuation. This approach allows one to compute a near optimal policy in a rather simple and relatively fast fashion, for example, by sequentially reoptimizing the I model, which is a deterministic and dynamic model that can be optimized very efficiently. Reoptimizing the ADP model, which is a stochastic and dynamic model, yields a slightly better policy at the expense of significantly higher CPU requirements. Reoptimizing the LP model, which is a stochastic and static model, also generates a very good policy and is faster than reoptimizing the ADP model.

6.7 Conclusions

The valuation of natural gas storage contracts is an important problem in practice. Exact valuation of these assets using the multidimensional models of the evolution of the natural gas futures curve that seem to be used in practice is an intractable problem. Thus, practitioners typically value such assets using heuristics. This chapter discusses an ADP approach to benchmark the effectiveness of such heuristics on realistic instances and possibly improve on their valuation performance. The ADP approach yields both lower and upper bounds on the value of storage using the multidimensional representation of the dynamics of the natural gas futures curve.

The upper bounds estimated using this approach appear to be fairly tight. The analyzed practice-based heuristic policies are very fast to compute but also substantially suboptimal, and are dominated by the ADP policy. However, the reoptimization-based versions of all of these policies are nearly optimal. The price to be paid for this improvement
is a significantly higher computational burden. Overall, sequential reopti-

mization, within Monte Carlo simulation of the futures curve, of the
deterministic model that computes the intrinsic value of storage
strikes a good balance between valuation quality and computational
requirement. However, in some cases reoptimization of the ADP model
can improve on the performance of the reoptimized intrinsic model, but
requires more time.

These results have immediate relevance for managers of natural gas
storage assets. The findings about reoptimization remain substantially
similar when (artificially) removing the seasonality from the employed
natural gas futures curves. This observation suggests that these results
have also potential relevance for managers involved in the valuation
of storage assets for other commodities, whose futures curves do not
exhibit the marked seasonality of the natural gas futures curve.

6.8 Notes

This chapter is based on Lai et al. [146], whose paper includes the
proofs of the results stated here. As mentioned in §5.8 in Chapter 5, the
notation used in this chapter is not fully consistent with the notation
of Lai et al. [146], but it is consistent with the notation of Chapter 5.

In their study of the effect of futures price term structure model
error on merchant commodity storage, Secomandi et al. [194] also in-
vestigate the structure of an optimal policy for the SDP formulated
in §6.2. They show that such a policy is of the basestock type, with
basestock targets that in each stage depend on the entire futures curve.
This structure extends the one presented in §5.3 in Chapter 5.

Eydeland and Wolyniec [86, p. 362] and Gray and Khandelwal [111]
discuss versions of the practice-based policies studied in this chapter.
Specifically, Eydeland and Wolyniec [86, p. 362] present the LP pol-
icy, and Gray and Khandelwal [112] consider the I, RI, LP, and RLP
policies. Gray and Khandelwal [112] refer to the RI policy as the rolling-
intrinsic policy, and the LP and RLP policies as the static and dynamic
basket-of-spreads policies.

Bjerksund et al. [19] and Wu et al. [231] also analyze the perform-
ance of the rolling intrinsic policy. Secomandi [192] investigates struc-
tural properties of the intrinsic and basket-of-spreads policies and their rolling versions, and provides theoretical support for the benefit of reoptimization observed in this chapter. He also conducts a more extensive and up-to-date numerical investigation of these policies, including a variant of the LP policy.

Nadarajah et al. [165] obtain stronger dual upper bounds for the instances considered in this chapter. They show that on the Winter instances the dual upper bound used in this chapter is between 2.46% and 2.82% larger than theirs. This finding implies that the RI, RLP, and RADP policies are near optimal also on these instances.

The heuristics considered in this chapter are examples of control algorithms, which are optimization-based models that compute heuristic control policies for intractable MDPs (Secomandi [188]). Other ADP approaches include Monte Carlo simulation and regression (Longstaff and Schwartz [154], Bertsekas and Tsitsiklis [16]), Monte Carlo simulation and math programming (Powell [174]), rollout policies (Bertsekas [15, Chapter 6]), and Monte Carlo simulation and optimization algorithms (Chang et al. [52]). Boogert and de Jong [25, 26], Carmona and Ludkovski [46], Nascimento and Powell [168], and Nadarajah et al. [164] value natural gas storage using ADP methods based on Monte Carlo simulation. In particular, Nadarajah et al. [165] establish that the ADP model studied in this chapter is a relaxation of an approximate linear program and propose additional approximate linear programming relaxations (see de Farias and Van Roy [68] and Adelman [1] for discussions of approximate linear programming). Boogert and Maziéres [27] and Thompson [209] apply radial basis function techniques to value natural gas storage. Felix and Weber [92] approach the same problem using recombining trees. Desai et al. [75] develop a pathwise optimization approach to the valuation of Bermudan options. Bonnans et al. [23] develop a stochastic dual dynamic programming approach to value energy contracts.

The upper bounds used in this chapter are based on the theory developed by Brown et al. [39], who generalize the work of Davis and Karatzas [67], Rogers [178], Andersen and Broadie [4], and Haugh and Kogan [117] on pricing American options (see Glasserman [105, Chapter 8] for a review). Recent work includes the papers by Belomestny et
al. [11], Bender [12], Chandramouli and Haugh [51], and Meinshausen and Hambly [161]. As mentioned, Nadarajah et al. [165] estimate dual upper bounds on the value of natural gas storage. Moreover, Secomandi [192] proposes a simplified dual upper bounding approach for commodity storage and shows its effectiveness on a set of natural gas instances.
This brief chapter considers the optimal management of commodity conversion assets other than storage, a topic introduced in §7.1. An inventory disposal asset is modeled in §7.2, an inventory acquisition asset in §7.3, and swing assets in §7.4. Section 7.5 offers a few concluding remarks. Section 7.6 includes examples of inventory disposal and acquisition assets, in addition to the ones in §§7.2-7.3, and gives pointers to the literature.

7.1 Introduction

The commodity conversion assets considered in this chapter are related to the commodity storage assets studied in Chapter 5. Inventory disposal/acquisition assets are similar to storage assets, except that they are one-sided with only sale decisions for inventory disposal assets and only purchase decisions for inventory acquisition assets. Another difference from the storage asset is that they have also constraints on the amount of inventory that must be disposed-of/acquired by a given date. This implies that the feasible inventory set is stage dependent. Swing assets also are akin to storage assets, but their payoffs represent
savings/gains from the contractual rights to buy/sell a commodity at contractual prices that differ from market prices.

The results in this chapter illustrate how the basestock structure that characterizes the optimal inventory trading policy for commodity storage assets can be modified for the optimal management of inventory disposal/acquisition and swing assets. Thus, the formulations of these problems as MDPs and their analysis rely substantially on the formulation of the storage asset as an MDP and the analysis carried out in Chapter 5. In addition, this chapter emphasizes that such modified basestock structures share the same complexity of the basestock structure for storage assets, as well as, under natural assumptions on the relevant operational parameters, its lot size characterization stated in part (b) of Proposition 4 in §5.5 in Chapter 5.

### 7.2 An Inventory Disposal Asset

Consider a merchant who has acquired a given amount of a storable commodity and wants to unwind this position in a fixed time horizon. For instance, this might be a merchant of a particular metal, such as aluminum, whose inventory is stored in a warehouse on its account (Geman [101, Chapter 8]). Sales from inventory are marked at the prevailing spot price when each sale occurs. There is a limit on the amount of commodity that the merchant can sell per unit of time, e.g., due to the warehouse operational constraints. This merchant thus owns an inventory disposal asset with an embedded timing option about when to sell.

Different from the storage asset studied in Chapter 5, for which there is no constraint on the amount of inventory to be held at the end of the planning horizon, the manager of an inventory disposal asset must sell all of the initial inventory by the end of the planning horizon. Hence, the terminal inventory level is constrained to be zero.

The problem of managing an inventory disposal asset can be formulated as an MDP. The time horizon is \([0,T]\). Inventory disposal decisions are made at each time \(T_i\), with \(i \in \mathcal{I} := \{0, \ldots, N - 1\}\), \(T_0 := 0\), and \(T_N := T\). The stage set is \(\mathcal{I}\). The initial inventory is \(\bar{x}\). The maximum amount of inventory that can be sold in each stage is
the positive number $C$, which is less than or equal to $\bar{x}$. The feasible inventory set in stage $i$ is $X_i := [0, \bar{x} \land (N - i)C]$, since the initial inventory must be entirely sold off by time $T$ and the capacity $C$ limits how much inventory can be sold in each stage.

Selling an amount of inventory $a$, a nonnegative quantity, in stage $i$ generates the cash flow $(s_i - c)a$, where $c$ is a marginal operational cost. A holding cost $h$ is charged against each unit of inventory available in a given stage.

Given a feasible inventory level $x$ in stage $i$, a feasible action $a$ must be nonnegative, cannot exceed the selling capacity $C$, and must result in a feasible inventory level in the next stage, that is, a value in set $X_{i+1}$:

$$0 \leq a \leq C,$$

$$0 \leq x - a \leq \bar{x} \land ((N - i - 1)C).$$

The resulting feasible action set is

$$A_i(x) := [(x - (\bar{x} \land (N - i - 1)C)) \lor 0, x \land C].$$

For simplicity, assume that the spot price follows a one-factor process in which the current spot price is a sufficient statistic for the distribution of spot prices in later stages (see §4.2 in Chapter 4). This assumption can be easily relaxed for the purposes of the analysis performed in this section. As in §5.2 in Chapter 5, let $A_i^\pi(x_i, s_i)$ be the decision rule of policy $\pi$ in stage $i$, $\Pi$ the set of all the feasible policies, and $\tilde{x}_i^\pi$ the random inventory level available in stage $i$ when following policy $\pi$. Denote the (constant) per stage risk-free discount factor by $\delta$ and risk-neutral expectation by $E^{RN}$.

The initial (stage 0) inventory level is $\bar{x}$. The objective is to solve the following optimization model:

$$\max_{\pi \in \Pi} \sum_{i=0}^{N-1} \delta^i E^{RN} [(\tilde{s}_i - c) A_i^\pi(\tilde{x}_i^\pi, \tilde{s}_i) - h\tilde{x}_i^\pi | x_0 = \bar{x}, s_0].$$  \hspace{1cm} (7.1)

To understand the structure of the optimal policy of model (7.1), ignore for a moment the requirement that the initial inventory must be entirely sold by time $T$. In this case the inventory disposal asset is
analogous to the storage asset discussed in Chapter 5 with respective injection and withdrawal capacities equal to 0 and \(-C\), marginal injection and withdrawal costs equal to infinity and \(c\), and fuel loss factors equal to 1. (The minus sign in front of \(C\) here is due to the withdrawal action and capacity being modeled as negative quantities in Chapter 5.) It thus follows from the analysis of Chapter 5 that the optimal policy for such an inventory disposal asset has a basestock target structure. Specifically, in every stage and for a given spot price realization, the buy-up-to basestock target can be defined to be zero and the only relevant basestock target is the sell-down-to target. In every stage and for a given spot price, the feasible inventory level is partitioned into at most two regions: a do-nothing region for inventory levels below the sell-down-to basestock target and a selling region for inventory levels above this target.

This discussion suggests that adding the constraint that the entire initial inventory be sold by the end of the planning horizon should not fundamentally change the optimal policy structure. Proposition 8 formally states that this structure is of the single basestock target type.

**Proposition 8 (Inventory disposal asset).** The optimal policy for model (7.1) is of the basestock target type. In every stage \(i \in I\) given the spot price \(s_i \in \mathbb{R}_+\) there exists a critical inventory level \(b_i(s_i) \in \mathcal{X}_i\) such that an optimal action in state \((x_i, s_i) \in \mathcal{X}_i \times \mathbb{R}_+\) is \(0 \lor ((x_i - b_i(s_i)) \land C)\).

Proposition 8 relies on the basic intuition behind the basestock structure for storage assets established in Theorem 5.2 in Chapter 5. In a given stage and for a given spot price, the continuation value function of the formulation of model (7.1) as an SDP, which is not shown here for brevity, is concave in inventory. The linearity of the immediate payoff function in the amount of sold inventory then implies the stated basestock target structure.

Analogous to storage assets, the computation of an optimal inventory disposal policy is greatly simplified by the capacity \(C\) and the maximum inventory level \(\bar{x}\) being integer multiples of a given lot size \(Q\). Indeed, in this case each basestock target also can be taken to be an
integer multiple of $Q$. That is, the basestock target structure has a lot size characterization.

Example 1 in §5.3.2 in Chapter 5 shows the general complexity of the basestock target structure for a slow storage asset; namely, that the optimal policy for a slow storage asset is not of the bang-bang type. The same is true for the optimal policy of a slow ($C < \pi$) inventory disposal asset. This feature can be seen by tailoring this example to such an asset.

Specifically, consider the medium, low, and high deterministic spot price path illustrated in Figure 5.5 in §5.3.2 of Chapter 5. Set the total inventory to be sold $\pi$ equal to 1, the discount factor $\delta$ equal to 1, the selling capacity $C$ equal to $2/3$, and the marginal cost $c$ and the holding cost $h$ equal to 0. This means that the lot size $Q$ is equal to $1/3$. Given that the initial inventory level is 1, it is clearly optimal to sell $1/3$ of it in stage 0 and $2/3$ of it in stage 2. This means that the basestock targets are $2/3$ in stage 0, 1 in stage 1, and 0 in stage 2. The selling capacity is thus optimally underutilized when selling in stage 0.

Similar to Example 1 in §5.3.2 in Chapter 5 that brings to light the notion of left over space for a slow storage asset, this example illustrates that left over inventory is a general feature of a basestock target structure with a slow inventory disposal asset. The inventory left over is $1/3$, which is the difference between the initial inventory to be sold and the selling capacity per stage: $\pi - C = 1 - 2/3$.

### 7.3 An Inventory Acquisition Asset

An inventory acquisition asset is the flip-side of an inventory disposal asset. It represents a situation in which a merchant has agreed to deliver a given amount of commodity by a certain date, and thus must acquire this amount during a given time horizon. In doing so, the merchant has the flexibility (timing option) of choosing when to purchase the commodity from the spot market, holding it until the delivery date.

This problem can be formulated as an MDP. The time horizon is $[0, T]$. The merchant must have available an amount $\pi$ of commodity at time $T$. Purchases can be made at each of a given number of dates $T_i$ in set $[0, T]$, with $i \in I$ (this set is defined as in §7.2). The stage set
Due to operational constraints, there is a capacity \( C \) (a positive number less than or equal to \( \bar{x} \)) that limits the amount of commodity that can be purchased at each such date. The cash flow associated with the purchase of an amount of commodity \( a \) in stage \( i \) is \(- (s_i + c) a\), where \( c \) is a marginal operational cost. The holding cost \( h \) is assessed on each inventory unit available in a given stage.

The requirement on the amount of inventory to be procured by time \( T_N \equiv T \) and the limit on the amount that can be purchased at each time \( T_i \) imply that the feasible inventory set in each stage \( i \) is \( \mathcal{X}_i := [0 \vee (\bar{x} - (N - i) C), \bar{x}] \). A feasible purchase in stage \( i \) when the available inventory level is \( x \in \mathcal{X}_i \) must satisfy

\[
0 \leq a \leq C,
\]

\[
0 \vee (\bar{x} - (N - i - 1) C) \leq x + a \leq \bar{x}.
\]

These inequalities imply that the corresponding set of feasible purchases is

\[
\mathcal{A}_i(x) := [0 \vee ((0 \vee (\bar{x} - (N - i - 1) C)) - x), (\bar{x} - x) \wedge C].
\]

Like the inventory disposal asset, suppose for simplicity that the distribution of the spot price in the next stage only depends on the spot price in the current stage. The initial inventory level is 0. Reusing the notation used to formulate model (7.1), the optimization model to be solved is

\[
\min_{\pi \in \Pi} \sum_{i=0}^{N-1} \delta^i E^{RN} \left[ (s_i + c) A_i^\pi (\tilde{x}_i^\pi; \tilde{s}_i) + h \tilde{x}_i^\pi \mid x_0 = 0, s_0 \right]. \tag{7.2}
\]

As with the inventory disposal asset, the inventory acquisition asset is related to the storage asset, but the requirement that a given amount of commodity be procured by a given date is critical to make the problem nontrivial. In other words, if there were no such requirement, the optimal policy would be trivially equal to doing nothing in every stage for every spot price realization. In contrast, this is not the case for the inventory disposal asset when it lacks the requirement that the entire initial inventory be sold by the end of the time horizon.

Proposition 9 states that the optimal inventory acquisition policy for model (7.2) has a basestock target structure.
Proposition 9 (Inventory acquisition asset). The optimal policy for model (7.2) is of the basestock target type. In every stage $i \in I$ given a spot price $s_i \in \mathbb{R}_+$ there exists a critical inventory level $b_i(s_i) \in X_i$ such that an optimal action in state $(x_i, s_i) \in X_i \times \mathbb{R}_+$ is $0 \lor ((b_i(s_i) - x_i) \land C)$.

As with Proposition 8, the intuition behind Proposition 9 is the concavity in inventory of the continuation value function of the SDP formulation of model (7.2), not shown here for brevity, in each given stage and for a given spot price, and the linearity in the amount of purchased inventory of the immediate payoff function. Moreover, the basestock structure for an inventory acquisition asset shares the operational complexity and lot size characterization of the optimal policy for an inventory disposal asset.

7.4 Swing Assets

Swing contracts are widespread in energy, especially electricity and natural gas, industries. Energy producers or buyers often transact via contracts that specify minimum and maximum total amounts of energy to be sold or purchased at a fixed price during a given time period. Moreover, these contracts specify a maximum, and possibly a minimum, amount of energy transacted on each given trading date. Thus, a producer or a buyer with such a contract is obligated to deliver or purchase at least a given amount of energy, but retains the flexibility to decide how to do so during the contractual time period. In other words, the owners of such contracts have available a given number of “swings” and must decide how to optimally exercise them over the contract life. The distinction here is between sale-swing and purchase-swing contracts.

The availability of energy spot markets means that producers or buyers could transact in such markets at the prevailing market price, rather than through a given swing contract. Thus, the valuation and management of swing contracts involve optimally managing the gains/savings that accrue to the owner of a swing contract relative to trading in the spot market. Consequently, the dynamics of energy
7.4. Swing Assets

Spot prices are fundamental in the valuation and management of swing contracts.

Sale-swing asset. Consider an energy producer that has agreed to deliver at least $\underline{x}$ but no more than $\overline{x}$ units of energy during the time interval $[0,T]$ to a given buyer. Sales can be made at each of a given number of dates $T_i$ with $i \in I$ (the same set used in §7.2). Each sale can be made at a contractually fixed unit price equal to $K$. There is a limit $C$, a positive number, on the amount of each sale (the possibility of a minimum sale quantity per date is ignored for simplicity).

If a sale $a$ is made at time $T_i$ via the sale-swing contract, then the producer gains the amount $(K - s_i)a$. This gain arises because the producer could have sold the amount of energy $a$ on the spot market for a cash flow equal to $s_i a$, but ownership of the sale-swing contract gives the producer the ability to obtain additional value relative to transacting on the spot market.\(^1\) This is a key insight that allows the problem of managing a sale-swing contract to be formulated as an MDP.

The stage set of this MDP is the set $I$. The total amount of energy sold since time 0 by time $T_i$ is denoted by $x_i$ and represents the amount of energy already sold up to stage $i$, a type of “inventory.” The feasible inventory set in stage $i$ is $X_i := [0 \lor (\underline{x} - (N - i)C), \overline{x}]$.

Given the feasible inventory level $x$ in stage $i$, a feasible action must satisfy the constraints

$$0 \leq a \leq C,$$

$$0 \lor (\underline{x} - (N - i - 1)C) \leq x + a \leq \overline{x}.$$  

These inequalities imply that this action must belong to the set

$$A_i(x) := [0 \lor (x - (0 \lor (\underline{x} - (N - i - 1)C))), (\overline{x} - x) \land C].$$

As in §§7.2-7.3, assume for simplicity that the current spot price is sufficient for determining the future spot price dynamics. The initial

\(^1\)An implicit assumption here is that the operational cost of executing the sale is the same under both modes of operations.
inventory level is 0. Reusing the notation used in §§7.2-7.3, optimally managing a sale-swing asset entails solving the optimization model

\[
\max_{\pi \in \Pi} \sum_{i=0}^{N-1} \delta^i E^{RN} \left[ \left( K - \tilde{s}_i \right) A_{i}^{\pi}(\tilde{x}_{i}^{\pi}, \tilde{s}_{i}) \mid x_0 = 0, s_0 \right].
\] (7.3)

Model (7.3) resembles model (7.1) but with the key difference that its focus is on swing sales, which are more attractive when the spot price is low, rather than on spot sales. Notwithstanding the absence of the holding cost component in model (7.3) and some differences in the definition of the feasible inventory and action sets of these models, this observation suggests that the optimal policy for a sale-swing asset is of the basestock target type. Proposition 10 states that this is indeed the case.

**Proposition 10 (Sale-swing asset).** The optimal policy for model (7.3) is of the basestock target type. In every stage \(i \in I\) given a spot price \(s_i\) there exists a critical inventory level \(b_i(s_i) \in X_i\) such that an optimal action in state \((x_i, s_i) \in X_i \times \mathbb{R}_+\) is 0 \(\lor ((b_i(s_i) - x_i) \land C)\). Proposition 10 relies on the same intuition that underlies Propositions 8 and 9 in this chapter. Moreover, the optimal policy of a sale-swing asset shares the same complexity and lot size characterization of the optimal policies of the inventory disposal and acquisition assets (the lot size characterization holds under the assumption that the quantities \(x, \pi, \), and \(C\) are all integer multiples of some number \(Q\)).

**Purchase-swing asset.** A purchase-swing asset is the analogue of a sale-swing asset for an energy buyer. This buyer has agreed to purchase at least \(x\) but no more than \(x\) units of energy during the time interval \([0, T]\) from a given producer. On each of a given number of dates \(T_i\) with \(i \in I\), this buyer can purchase energy up to the limit \(C\) at a given unit price \(K\). The difference between the sale-swing asset and the purchase-swing asset is the benefit obtained by the buyer when a purchase is made. This per unit gain on a purchase made at time \(T_i\) is \((s_i - K)\), and represents the savings obtained by the buyer when purchasing the commodity using the purchase-swing asset rather than
the spot market. The total amount of energy purchased by date $T_i$ is denoted by $x_i$. Replacing the term $(K - s_i)$ with this per unit gain in model (7.3) is the only modification that needs to be made to this model to formulate the optimal management of the purchase-swing asset as an MDP. It follows that the analogue of Proposition 10 in this section and the properties discussed after this result hold for the purchase-swing asset.

### 7.5 Conclusions

This chapter considers the management of inventory disposal/acquisition assets and swing assets. Inventory disposal/acquisition assets are relevant for a merchant who must sell/procure a given amount of inventory by a certain date, and has the flexibility to decide when to sell/purchase this inventory. Swing assets are widespread in the energy industry and provide sale/purchase-mode flexibility to energy producers and users.

Variants of the basestock target structure that characterize the optimal management of storage assets are also optimal for the management of the conversion assets considered here. Moreover, these variants of the basestock target structure retain the operational complexity and the lot size characterization of the storage basestock target structure.

### 7.6 Notes

The inventory disposal asset without the condition that the entire initial inventory be sold by the end of the finite horizon represents a natural resource production asset, e.g., a natural gas or petroleum reservoir or a coal mine; that is, the initial inventory represents the amount of available natural resource, e.g., oil, natural gas, or coal (see §2.4 in Chapter 2 for a description of a simplified version of such an asset). In such cases the dynamics of the natural resource availability may deserve more detailed modeling. For instance, after an initial transitory period, the natural gas or petroleum that can be extracted from a reser-

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2 This statement implicitly assumes an equal operational cost of performing the purchase under both modes of operations.
voir in a given time period has been observed to exhibit exponential decline as the total natural gas or petroleum in the reservoir depletes. Such decline may also be characterized by stochastic behavior. Smith and McCardle [200] and Enders et al. [84], among others, model oil and natural gas production assets as SDPs. Brown and Smith [38] take a bandit approach to the management of oil and gas production assets. Cortazar et al. [61] apply the least squares Monte Carlo approach (Longstaff and Schwartz [154]) to value a copper mine.

The inventory acquisition asset can be modified to a consumption asset (see §2.4 in Chapter 2), whereby the amount of acquired commodity is used as an input to a manufacturing or distribution stage. The required amount of commodity may not be known with certainty, that is, the demand for the commodity may be random, and excess supply can be carried in inventory to satisfy demand in later periods. Nascimento and Powell [167] and Secomandi and Kekre [193] discuss models in which the commodity/energy requirement is stochastic (in the model of Secomandi and Kekre [193] energy purchases can be made both in spot and forward markets incurring differential transaction costs). Kalymon [129], Gavirneni [100], Goel and Gutierrez [106, 107], Berling and Victor Martínez-de-Albéniz [14], and Kouvelis et al. [142, 143] present commodity inventory management models with stochastic demand and purchase prices (Goel and Gutierrez [107] do this in a multiechelon setting).

This chapter does not deal with cross-commodity conversion assets that involve the physical conversion of one commodity into one or more commodities. Examples include refinery assets that convert petroleum into gasoline, naphtha, and jet fuel, or corn or sugarcane into ethanol, or soybean into soyoil and soymeal; power plants that use natural gas; and the processing of cattle into beef products. Power plants are often subject to pollution emission caps which limit their operations. Operation of a power plant then involves choosing which days will be the most profitable – given the spark spread between power and fuel prices – to produce power and use up the available inventory of emission permits. This decision has a similar timing option structure as an inventory disposal asset. If emission permits are tradable, as with SOX in the United States or carbon in Europe, then the operation of a power plant is a
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variation on a storage asset in which an inventory of emission permits can be bought, sold, and also used over time.

Coal and natural gas fired power plants also have access to storage of their fuels. In the case of natural gas, in addition to underground natural gas storage facilities, the pipeline linepack can also be used for short-term storage. LNG terminals also provide short-term storage opportunities. In particular, the short-term planning of sales from an LNG regasification terminal to make space for an incoming shipment can be modeled as an inventory disposal asset.

Fleten and Kristoffersen [96], Näskäkälä and Keppo [166], and Denault et al. [73] consider hydropower generation assets. The short-term management of these assets is related to the management of inventory disposal assets. However, in this case it might not be desirable to entirely drain the reservoir that stores the water used for power generation. Modeling this aspect amounts to redefining the terminal inventory level to be strictly positive. Moreover, random water inflows might occur during the planning horizon. Relevant work in the area of hydropower production, but not from a real option perspective, includes that of Drouin et al. [81] and Lamond et al. [148].

The management of windpower generation assets and LNG production and regasification assets are closely related topics. Kim and Powell [139] and Zhou et al. [233], among others, investigate the value of storage for a wind-based power generator. Lai et al. [147] consider LNG storage by modeling the shipping, storage, and regasification of LNG. Guigues et al. [113] develop a stochastic programming approach to manage LNG contracts modeling storage and cancellation provisions. Wallace and Fleten [220] review the related literature on energy stochastic optimization models.

Research on the management of cross-commodity conversion assets includes the work of Adkins and Paxson [2], Arvesen et al. [5], Boyabatli et al. [30], Brandão et al. [33], Devalkar et al. [76], Dockendorf and Paxson [79], Hahn and Dyer [115], Kazaz and Webster [134], Thompson et al. [211], Tseng and Barz [217], Tseng and Lin [218], Wu and Chen [230], Plambeck and Taylor [172], and Thompson [210].

The swing assets considered in this chapter are partially based on the description of swing contracts in Geman [101, p. 294]. The insight
that a swing contract can be valued and managed relative to the strategy that only operates in the spot market appears novel. Ghiuvea et al. [103], Jaillet et al. [125], Keppo [137], Barrera-Esteve et al. [9], Ross and Zhu [181], and Nadarajah et al. [164], among others, discuss the valuation and management of swing contracts.
This chapter briefly discusses a nonexhaustive list of trends for further research in the area of merchant operations. The topics considered are financial hedging in §8.1; the analysis of other commodity conversion assets in §8.2; approximate dynamic programming methods in §8.3; price model error in §8.4; endogeneity of the price process in determining an operating policy in §8.5; equilibrium asset pricing in §8.6; and the choice of capacity levels in §8.7.

8.1 Financial Hedging

In addition to being one of the basic principles underlying risk-neutral valuation in dynamically complete markets, the ability to construct dynamic portfolios of traded securities (typically, commodity and energy futures contracts) that replicate the cash flows generated by commodity conversion assets is an important risk management tool (Hull [122]). Merchants can use these replicating portfolios for financial hedging.

Specifically, shorting and periodically rebalancing a replicating portfolio can offset the changes over time of the market value of a given asset. This approach yields a financial hedging policy, known as
delta hedging (see §3.3 and expression (3.11)). Delta hedging can be used to reduce/eliminate risk capital charges that a merchant might incur in some costly states of nature (see, e.g., Tirole [213, §5.4] and references therein).

Due to the lack of explicit representations for optimal, or near-optimal, operating policies for most commodity conversion assets, closed-form expressions for the deltas and other hedging parameters, collectively called the Greeks, are rare. Numerical computation of the Greeks is thus the norm. Monte Carlo simulation is a useful tool in this respect. Boyle et al. [31], Broadie and Glasserman [37], and Fu and Hu [99] discuss the estimation of the Greeks by Monte Carlo simulation. Glasserman [105, Chapter 7] provides a more recent review of this literature. Wang et al. [222] is a recent addition to this literature. Kaniel et al. [133], Chen and Liu [57], and Caramellino and Zanette [43] focus on the estimation of deltas for American options.

Secomandi and Wang [195] and Secomandi et al. [194] use derivative estimation techniques to compute the deltas of natural gas transport and storage assets. Bonnans et al. [24] compute the price sensitivities of the values of energy contracts in a stochastic programming setting. Fleten and Wallace [97] and Li and Kleindorfer [150] consider the delta hedging of hydropower generation assets and spark spread options.

More research is needed to benchmark the performance of the financial hedging in realistic situations. Research on hedging is particularly important in dynamically incomplete markets, for which risk-neutral valuation depends on price-of-risk assumptions.

8.2 Analysis of Other Commodity Conversion Assets

This monograph examines in detail only a limited number of applications: storage and inventory disposal/acquisition/swing assets. As discussed in §2.6 in Chapter 2 and in §7.6 in Chapter 7, the ideas and methods of merchant operations have much wider applicability beyond these specific applications. Important commodity conversion assets that have received substantial attention in the literature, but are ignored in

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1 In addition, Secomandi and Wang [195] consider other Greeks and Secomandi et al. [194] test their approach on natural gas data.
this monograph, include power plants and a variety of commodity processing assets that embed portfolios of spread/rainbow and switching options.

The structure of the optimal operating policies of these assets is typically not well understood in the literature. In particular, unlike the optimal policy of the assets that are the focus of this monograph, the optimal policy structure for these other assets is unlikely to be of the basestock type. Deepening our understanding of the structure of their optimal policies is a fruitful area for additional research. Examples include optimal policies given ramp-up/cool-down requirements, start-up/shut-down costs, and path-contingent operating performance. Although the computation of optimal policies is typically an intractable task, knowledge of these structures might inform the development of practical and effective methods for the computation of near-optimal policies.

8.3 Approximate Dynamic Programming

As illustrated in Chapter 6, practitioners use high-dimensional commodity price evolution models. Even though one may not want to use a full-dimensional market model of the futures curve evolution, a model with three factors would already be considered high dimensional for the exact computation of optimal operating policies. Both practitioners and scholars have thus turned to approximations that allow for the efficient computations of near-optimal operating policies.

Due to the dynamic and stochastic nature of most merchant operations problems, these approximations involve some form of approximate dynamic programming algorithm. Glasserman [105, Chapter 7] discusses the literature on the Monte Carlo pricing of American options, including approximate dynamic programming methods. This material is relevant to merchant operations problems, which tend to have more complicated structures than American or Bermudan options. Moreover, as discussed in §6.8 in Chapter 6, the literature includes several approximate dynamic programming methods. Refining these methods via their application to merchant operations problems is an interesting area for additional research.
Approximate dynamic programming typically focuses on the computation of near optimal policies and bounds to stochastic and dynamic optimization problems. As demonstrated in Chapter 6, good bounds are essential to assess the performance of heuristic policies. As discussed in §6.8 in Chapter 6, there is an active stream of recent research that deals with bound estimation in financial and real options valuation. Continuing this line of work in the area of merchant operations, in which optimization problems are more complicated than stopping problems, is a promising area for additional research.

8.4 Price Model Error

Merchant operations relies in a fundamental manner on models of the evolution of commodity and energy prices. It is inevitable that these models are only approximations of the dynamics of these prices. In other words, these models embed errors. It is thus important to assess the impact of model errors on the merchant management of different commodity conversion assets. The impact can be on the asset valuation (which merchants use to decide how much to bid to acquire the asset), the asset operating policy and, when relevant, the asset financial hedging policy. When the impact of model errors is practically important, developing methods to mitigate its negative consequences is relevant. This type of research thus combines empirical and methodological aspects. Secomandi et al. [194] investigate these issues in the context of commodity storage assets. It would be interesting to study price model error in the context of different commodity conversion assets.

8.5 Price Impact

A key underlying assumption in the analysis in this monograph is that a merchant’s trading decisions do not affect market prices. That is, the capacity of the commodity conversion asset under a merchant’s control is “small” relative to the market size or, equivalently, commodity markets are sufficiently liquid. In such a setting, market liquidity considerations do not play a role in optimizing the operating policy of a conversion asset, and a price-taker modeling approach is justified. How-
ever, liquidity in physical spot markets can be limited. In these cases, the price-taker approach is not justified and may generate suboptimal operating policies once liquidity constraints are taken into account. Studying the extent of this suboptimality and developing merchant operations methods in the presence of limited liquidity are interesting avenues for additional research. Some research along these directions includes the papers of Martínez-de-Albéniz and Vendrell Simón [159], Felix et al. [91], and Chaton and Durand-Viel [55].

8.6 Equilibrium Asset Pricing

Reduced-form models like the Pilipovic/Schwartz-Smith model or the future term structure factor models (see §4.2 and §4.3 in Chapter 4) simply take commodity price dynamics as exogenously given. As such, reduced-form models make strong assumptions about the stationarity of commodity price processes over time. In contrast, equilibrium models derive the dynamics of commodity prices from the statistical properties of random environmental shocks and the joint endogenous production, storage, and consumption behavior of optimizing producers, merchants, and consumers. By relating commodity prices to deeper economic fundamentals, equilibrium models have at least some hope of being able to account for changing statistical properties of commodity prices. Section 4.4 in Chapter 4 gives pointers to recent theoretical work on the relationship between investor preferences and commodity risk premia and to recent empirical work on the macroeconomic and microeconomic drivers of commodity prices. While option valuation theory is concerned with computing market prices, merchant traders frequently trade on views about future changes in fundamentals that differ from the prevailing market view. Commodity pricing when agents have strategic market power or private information and the impact of market segmentation between physical and financial markets for commodities are relatively unexplored topics that are not well understood.
8.7 Capacity Choice

This monograph assumes that the capacity of a commodity commodity conversion asset is given. That is, a merchant has already decided how to size the assets that it controls, e.g., how much space and injection/withdrawal capacity of a given natural gas storage asset to rent and for how long. This capacity choice problem is important in practice when a merchant faces risk capital charges, that is, when there are frictions in capital markets. In this situation, a merchant might not be able to fund all the available projects, so that a merchant might have to choose which projects to select and their size. The resulting problem might resemble a portfolio optimization problem, even though projects might be selected in a sequential manner rather than simultaneously. It would be interesting to learn how merchants currently address this project selection and sizing problem, and to investigate whether optimization techniques might be relevant to support merchants in solving such problem. This topic is also related to the capacity choice and sequential investment problems studied in the real option literature (see, e.g., Dixit and Pindyck [78]).
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References

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Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA, USA, 2012.


References


