On the Facets of Mixed Integer Programs with Two Integer Variables and Two Constraints

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Abstract. In this paper we consider an infinite relaxation of the mixed integer linear program with two integer variables and two constraints, and we give a complete characterization of its facets. We then derive an analogous characterization of the facets of the underlying finite integer program.

1 Introduction

We consider the mixed 2-integer-variable linear program with two constraints

$$\begin{array}{l} x = f + \sum_{j=1}^{k} r^j s_j \\ x \in \mathbb{Z}^2 \\ s \in \mathbb{R}^k_+ \end{array} \tag{1}$$

where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, $k \geq 1$, and $r^j \in \mathbb{Q}^2$. Let $R_f(r^1, \ldots, r^k)$ be the convex hull of all vectors $s \in \mathbb{R}^k_+$ such that $f + \sum_{j=1}^{k} r^j s_j$ is integral. $R_f(r^1, \ldots, r^k)$ is a polyhedron (We refer the reader to [11] for standard definitions). Model (1) was considered by Andersen, Louveaux, Weismantel and Wolsey [1]. They showed that the nontrivial facets of $R_f(r^1, \ldots, r^k)$ are necessarily defined by split inequalities or intersection cuts (Balas [2]) arising from triangles or quadrilaterals in $\mathbb{R}^2$. A goal of this paper is to give a converse to the result in [1]: which splits, triangles and quadrilaterals actually define facets of $R_f(r^1, \ldots, r^k)$?

Gomory and Johnson [8] suggested relaxing the k-dimensional space of variables $s = (s_1, \ldots, s_k)$ to an infinite-dimensional space, where the variables s_r are defined for any $r \in \mathbb{Q}^2$. We get the *infinite program with two integer variables and two constraints*

$$\begin{array}{l} x = f + \sum r s_r \\ x \in \mathbb{Z}^2 \\ s \geq 0 \quad \text{with finite support.} \end{array} \tag{2}$$

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The vector $s = (s_r)_{r\in\mathbb{Q}^2}$ is said to have *finite support* if $s_r \neq 0$ for a finite number of $r \in \mathbb{Q}^2$. Let R_f be the convex hull of all vectors $s \geq 0$ with finite support such that $f + \sum rs_r$ is integral. Note that the polyhedron $R_f(r^1, \dots, r^k)$ is the face of R_f obtained by setting $s_r = 0$ for all $r \in \mathbb{Q}^2 \setminus \{r^1, \dots, r^k\}$. The motivation for working with R_f instead of $R_f(r^1, \dots, r^k)$ is that it only has one parameter, namely f, and therefore the results are cleaner. Moreover results obtained for R_f can be carried over to the model used in [1].

We say that an inequality is *valid* for R_f (resp. $R_f(r^1, \dots, r^k)$) if it is satisfied by all vectors in R_f (resp. $R_f(r^1, \dots, r^k)$). Inequalities $s_i \geq 0$ are called *trivial* valid inequalities. In this paper, we discuss only nontrivial valid inequalities. The solution $s = 0$ is not feasible for R_f. Any valid inequality for R_f that cuts off the vector $s = 0$ is of the form

$$\sum \psi(r)s_r \geq 1 \tag{3}$$

where $\psi : \mathbb{Q}^2 \to \mathbb{R} \cup \{+\infty\}$ and, as above, we only consider vectors s with finite support. To avoid ambiguity, the product $+\infty \cdot 0$ is defined as 0.

Any valid inequality for R_f yields a valid inequality for $R_f(r^1, \dots, r^k)$ by simply restricting it to the space $r^1, \dots, r^k$. Furthermore, a full description of the polyhedron $R_f(r^1, \dots, r^k)$ is obtained from the set of valid inequalities for R_f by adding the constraints $s_r = 0$ for $r \neq r^1, \dots, r^k$. Therefore we will assume in the remainder that valid inequalities for $R_f(r^1, \dots, r^k)$ are restrictions of valid inequalities for R_f.

An inequality $\sum \psi(r)s_r \geq 1$ valid for R_f is *minimal* if there is no valid inequality $\sum \psi'(r)s_r \geq 1$ where $\psi' \leq \psi$ and $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{Q}^2$. We also say that such a function ψ is *minimal*. The following result was proved in [3].

Theorem 1. *A minimal valid function ψ is nonnegative homogeneous piecewise linear and convex. Furthermore, the closure of the set*

$$B_\psi := \{x \in \mathbb{Q}^2 : \ \psi(x - f) \leq 1\}. \tag{4}$$

is a full-dimensional polyhedron with 2, 3 or 4 edges, it contains no integral point in its interior but each edge contains an integral point in its relative interior.

A function ψ is *positively homogeneous* if $\psi(\lambda r) = \lambda\psi(r)$ for all $\lambda \geq 0$. Since ψ is always nonnegative in this paper, we simply say *homogeneous* to mean positively homogeneous. We will also simply say in the *interior* of an edge to mean in the *relative interior* of that edge.

The point f is in B_ψ since $\psi(0) = 0$. When f is in the interior of B_ψ, then ψ is continuous and B_ψ is closed (we call this the *nondegenerate* case), see Figure 1. In this case, the boundary of B_ψ is the set of points $x \in \mathbb{Q}^2$ that satisfy $\psi(x - f) = 1$. Thus, the knowledge of f and of the boundary of B_ψ together with the homogeneity of ψ is enough to compute the value of $\psi(r)$ for any vector $r \in \mathbb{Q}^2 \setminus \{0\}$: If $f + \lambda r$ is a point on the boundary of B_ψ for some $\lambda > 0$, we get that $\psi(r) = 1/\lambda$. Otherwise, if there is no such λ, r is an unbounded direction of

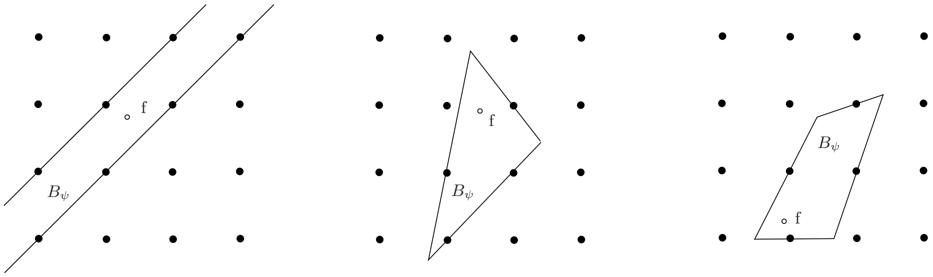


Fig. 1: Representation of B_ψ for nondegenerate cases.

B_ψ and $\psi(r) = 0$. We use the graphic representation of B_ψ to describe ψ when possible. The inequalities corresponding to the three cases of Figure 1 will be called *split*, *triangle* and *quadrilateral* inequalities. They are special case of the *intersection cuts* of Balas [2]. Solid lines in Figure 2 give level curves of $\psi(r)$ with values 0 and 1 for the three examples of Figure 1.

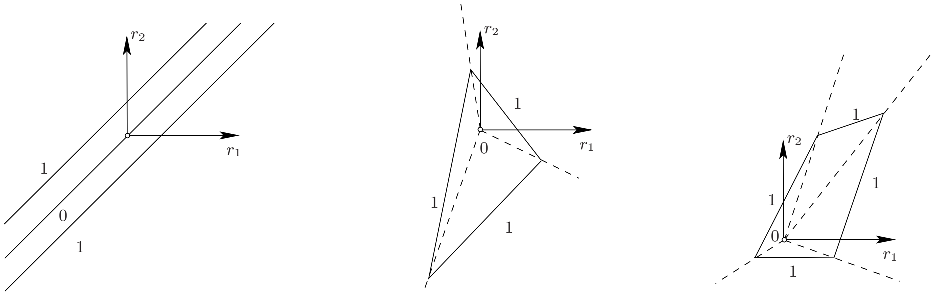


Fig. 2: Level curves of $\psi(r)$ for nondegenerate cases.

When f is a vertex of clB_ψ (the closure of B_ψ) or when f lies on one of its edges, ψ is neither continuous nor finite everywhere [3] (*degenerate case*), see Figure 3. In particular, for any direction $r \neq 0$ such that the half-line $L_r = \{x = f + \lambda r \text{ for } \lambda > 0\}$ is outside clB_ψ, we have $\psi(r) = +\infty$. For the directions such that the half-line L_r goes through the interior of clB_ψ, let $f + \lambda r$ be the point where L_r intersects the boundary of clB_ψ; then we get $\psi(r) = 1/\lambda$. Finally, when L_r supports an edge of clB_ψ, let $y = f + \lambda r$ be the first integral point encountered on L_r starting from f and let $x = f + \mu r$ be the first vertex of clB_ψ encountered (if any); if y is encountered first, we get $\psi(r) = 1/\lambda$ and if x is encountered first, we get $\psi(r) = 1/\mu$. There are five different degenerate inequalities, depending of the type of set clB_ψ and the position of f on its faces: *degenerate split*, *vertex-degenerate triangle*, *edge-degenerate triangle*, *vertex-degenerate quadrilateral* and *edge-degenerate quadrilateral* inequalities. Solid lines in Figure 4 give level curves of $\psi(r)$ with values 0 and 1 for the three examples of Figure 3.

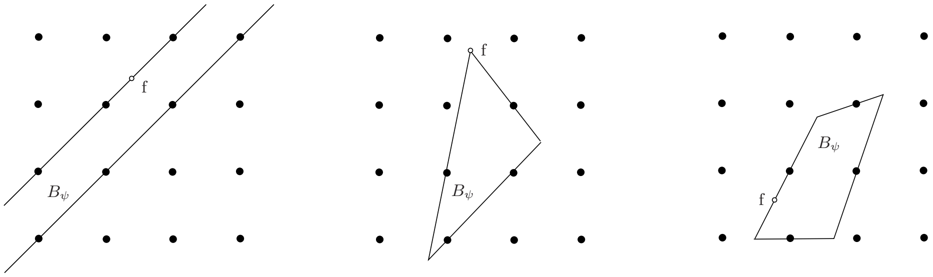


Fig. 3: Representation of B_ψ for degenerate cases.

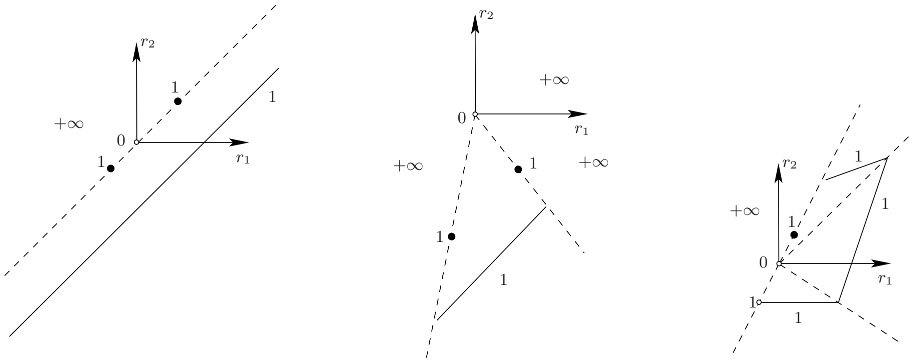


Fig. 4: Level curves of $\psi(r)$ for degenerate cases.

Note that Dey et al. [5] showed in a more general context that, if $\psi(r) < +\infty$ everywhere, then ψ is continuous, and therefore ψ is nondegenerate.

Polyhedra with no integral point in their interior but with an integral point in the relative interior of each facet are called *maximal lattice-free* [9]. The complete list of all maximal lattice-free convex sets in the plane is known:

Theorem 2. *[9] A maximal lattice-free convex set in the plane (x_1, x_2) is one of the following:*

i) An irrational line $ax_1 + bx_2 = c$, where a/b is irrational and $c \notin a\mathbb{Z} + b\mathbb{Z}$;
ii) A strip $c \leq ax_1 + bx_2 \leq c + 1$ where a and b are coprime integers and c is an integer;
iii) A triangle with an integral point in the interior of each of its edges;

iv) A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.

The polyhedra referred to in Theorem 1 correspond to the last three cases in Theorem 2. The first case does not play a role here as we only consider rational vectors f and r in the definition of R_f.

A valid inequality $\sum \psi(r) s_r \geq 1$ for R_f *defines a facet* of R_f if there does not exist two distinct valid inequalities $\sum \psi_j(r) s_r \geq 1$, $j = 1, 2$, such that $\psi = \frac{1}{2}\psi_1 + \frac{1}{2}\psi_2$. Note that, although we only use nontrivial inequalities in this definition, including them would give an equivalent definition. By extension, we also say that such a function ψ *defines a facet* of R_f. Gomory [7] recently raised the question of describing the facets of R_f. In this paper, we give a complete characterization of the facets of R_f.

A valid inequality $\sum_{i=1}^{k} \psi_j(r^i) s_i \geq 1$ for $R_f(r^1, \ldots, r^k)$ *defines a facet* of $R_f(r^1, \ldots, r^k)$ if two distinct valid inequalities $\sum_{i=1}^{k} \psi_j(r^i) s_i \geq 1$, $j = 1, 2$, do not exist such that $\psi(r^i) = \frac{1}{2}\psi_1(r^i) + \frac{1}{2}\psi_2(r^i)$ for $i = 1, \ldots, k$. This definition of a facet of $R_f(r^1, \ldots, r^k)$ is consistant with the usual definition of a facet of a polyhedron only if the polyhedron is full dimensional. The next lemma shows that this is the case.

Lemma 1. *If $R_f(r^1, \ldots, r^k)$ is non empty, then it is full dimensional.*

Proof. The recession cone of $R_f(r^1, \ldots, r^k)$ is $\mathbb{R}^k_+$.

The paper is organized as follows. Section 2 explains how results for the facets of R_f can be used to derive results for the facets of $R_f(r^1, \ldots, r^k)$. Section 3 shows that split inequalities are always facet defining for R_f, and that degenerate split inequalities also are. Section 4 deals with triangle and quadrilateral inequalities. It shows that triangle inequalities are always facets of R_f and it gives a necessary and sufficient condition for a quadrilateral inequality to define a facet. For the remaining degenerate cases, vertex-degenerate and edge-degenerate triangle inequalities are facet defining only if a condition on the integral points in the boundary of B_ψ is satisfied, whereas degenerate quadrilaterals never define a facet of R_f.

2 Facets of R_f and Facets of $R_f(r^1, \ldots, r^k)$

In this section, we give a relation between facets of R_f and facets of $R_f(r^1, \ldots, r^k)$. We first show that the degenerate cases can be ignored when dealing with $R_f(r^1, \ldots, r^k)$.

Theorem 3. *Let $r^1, \ldots, r^k$ be a set of $k \geq 1$ rays. Every nontrivial facet of $R_f(r^1, \ldots, r^k)$ can be obtained from a nondegenerate minimal valid function ψ for R_f.*

We will see in Theorem 6 that split inequalities always define facets of R_f. The situation for $R_f(r^1, \ldots, r^k)$ is a little bit more complicated, as the next theorem shows. See also Andersen et al. [1].

Theorem 4. *Let ψ be valid and minimal for R_f, with B_ψ unbounded and f in its interior. Let $r^1, \ldots, r^k$ be a set of $k \geq 1$ rays. Then ψ defines a split inequality $\sum_{i=1}^k \psi(r^i) s_{r^i} \geq 1$ for $R_f(r^1, \ldots, r^k)$. This inequality can also be obtained as a quadrilateral inequality if $\psi(r^i) > 0$ for $i = 1, \ldots, k$, and it defines a facet of $R_f(r^1, \ldots, r^k)$ otherwise.*

Nondegenerate minimal valid inequalities that are not split inequalities are generated by a function ψ with B_ψ bounded and f in the interior of B_ψ. Let $x^1, \ldots, x^k$ be the vertices of B_ψ. We always assume that these vertices are topologically ordered so that the edges of the boundary of B_ψ are convex combinations of x^i and x^{i+1} with indices taken modulo k. We define the *corner rays* of B_ψ to be the rays $\{r^1, \ldots, r^k\}$ joining f to the vertices of B_ψ, with $r^i = x^i - f$ for $i = 1, \ldots, k$.

The next theorem shows that if $r^1, \ldots, r^k$ are the corner rays of B_ψ then ψ is facet defining for R_f if and only if it is facet defining for $R_f(r^1, \ldots, r^k)$.

Theorem 5. *Assume that B_ψ is a polytope with f in its interior. Let ψ be valid and minimal for R_f and let $r^1, \ldots, r^k$ be the corner rays of B_ψ. Then ψ is facet defining for $R_f(r^1, \ldots, r^k)$ if and only if ψ is facet defining for R_f.*

The assumption of Theorem 5 that the rays are the corner rays of B_ψ can be relaxed slightly, keeping the proof almost identical:

Corollary 1. *Assume that B_ψ is a polytope with f in its interior. Let ψ be valid and minimal for R_f and let $r^1, \ldots, r^\ell$ be a set of rays including the corner rays of B_ψ. Then ψ is facet defining for $R_f(r^1, \ldots, r^\ell)$ if and only if ψ is facet defining for R_f.*

In [1], Andersen, Louveaux, Weismantel and Wolsey study $R_f(r^1, \ldots, r^k)$ and they prove that, when nonnegative combinations of $r^1, \ldots, r^k$ span $\mathbb{R}^2$, all the nontrivial facets of $R_f(r^1, \ldots, r^k)$ are split inequalities or are triangle or quadrilateral inequalities where the vertices of B_ψ are on the rays $f + \lambda r^i$, $\lambda > 0$, for $i = 1, \ldots, k$. They do not, however, describe precisely which triangles and quadrilaterals generate facets. Theorems 3, 4 and Corollary 1 show that the characterization obtained in this paper of the triangles and quadrilaterals that generate facets for R_f (Theorems 8 and 9) gives a complete characterization of the nontrivial facets of $R_f(r^1, \ldots, r^k)$.

3 Split Inequalities

3.1 Nondegenerate case

Consider a direction $r^0 \in \mathbb{Q}^2 \setminus \{0\}$ such that the line $L_0 := \{x = f + \alpha r^0,\ \alpha \in \mathbb{R}\}$ contains no integral point. Let L_1 and L_2 be parallel lines to L_0, each containing integral points, such that the set of points between L_1 and L_2 contains no integral point in its interior and contains L_0. (See Figure 5.) Define $\psi(r_0) = \psi(-r_0) = 0$, $\psi(x - f) = 1$ for any $x \in L_1 \cup L_2$. Since ψ is homogeneous, this defines $\psi(r)$ for all $r \in \mathbb{Q}^2$. The valid inequality $\sum \psi(r)s_r \geq 1$ is the well known *split inequality* [4]. These inequalities are equivalent to Gomory's mixed integer inequalities [6].

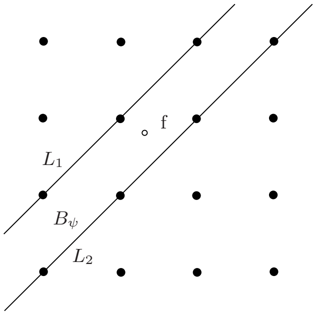


Fig. 5: Illustration for Theorem 6.

Theorem 6. *Split inequalities define facets of R_f.*

3.2 Degenerate case

Consider a direction $r^0 \in \mathbb{Q}^2 \setminus \{0\}$ such that the line $L_0 := \{x = f + \alpha r^0, \alpha \in \mathbb{R}\}$ contains integral points. Let L_1 be a line parallel to L_0 that contains integral points, such that the set of points between L_0 and L_1 contains no integral point in its interior. Let y^1 and y^2 be the first integral points encountered on the half-lines $f + \alpha r^0$, $\alpha \geq 0$, and $f - \alpha r^0$, $\alpha \geq 0$ respectively. (See Figure **??**.) Define $\psi(y^1 - f) = \psi(y^2 - f) = 1$ and $\psi(x - f) = 1$ for any $x \in L_1$. Since ψ is homogeneous, this defines $\psi(r)$ for all $r \in \mathbb{Q}^2$ in the closed half-space limited by L_0 and containing L_1. For all other $r \in \mathbb{Q}^2 \setminus \{0\}$, define $\psi(r) = +\infty$. The inequality $\sum \psi(r)s_r \geq 1$, a *degenerate split inequality*, is valid for R_f.

Theorem 7. *Degenerate split inequalities define facets of R_f.*

4 Triangle and Quadrilateral Inequalities

In this section, we assume that ψ is valid and minimal for R_f with f in the interior of B_ψ and that B_ψ is a polytope. Then by Theorem 1 and Theorem 2, B_ψ is either a triangle or a quadrilateral such that each of its boundary edges contains an integral point in its interior.

Let $x^1, \ldots, x^k$ be the vertices of B_ψ and $r^1, \ldots, r^k$ be the corner rays of B_ψ and let y^i be an integral point that can be obtained as a nontrivial convex combination of x^i and x^{i+1} for $i = 1, \ldots, k$ (indices are always implicitly taken modulo k).

Define M as the $2 \times k$ matrix whose column i is the vector y^i for $i = 1, \ldots, k$ (Recall that $k = 3$ or 4). Define X as the $2 \times k$ matrix whose column i is the vector x^i for $i = 1, \ldots, k$. Let S be the $k \times k$ matrix whose column i is the vector corresponding to the coefficients in the convex combination of x^i and x^{i+1} giving y^i for $i = 1, \ldots, k$.

We then have

$$M = X \cdot S \tag{5}$$

with

$$S = \begin{pmatrix} \alpha & 0 & 1-\gamma \\ 1-\alpha & \beta & 0 \\ 0 & 1-\beta & \gamma \end{pmatrix} \qquad \text{or} \qquad S = \begin{pmatrix} \alpha & 0 & 0 & 1-\delta \\ 1-\alpha & \beta & 0 & 0 \\ 0 & 1-\beta & \gamma & 0 \\ 0 & 0 & 1-\gamma & \delta \end{pmatrix}$$

where α, β, γ and δ are all strictly between 0 and 1.

Since we are interested in the dimension of faces of polyhedra, which requires checking affine independence of points, we add a third row full of 1s to the matrices M (resp. X) to obtain matrix $\bar{M}$ (resp., $\bar{X}$). Due to the specific form of the matrix S, we still have

$$\bar{M} = \bar{X} \cdot S \ . \tag{6}$$

Let A be an $m \times n$ matrix. The *nullspace* of A is $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ and the *columnspace* of A is $\mathcal{C}(A) = \{z \in \mathbb{R}^m \mid z = Ax \text{ for some } x \in \mathbb{R}^n\}$.

The following three results are classical results of linear algebra [10]:

Lemma 2. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then*

$$rank(A \cdot B) = rank(B) - dim(\mathcal{N}(A) \cap \mathcal{C}(B)) \ .$$

Corollary 2. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. If $rank(A) = n$, then*

$$rank(A \cdot B) = rank(B) \ .$$

Proof. If $rank(A) = n$, then $\mathcal{N}(A) = \{0\}$ and has dimension 0. Applying Lemma 2 yields the result.

Corollary 3. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then*

$$rank(A \cdot B) \leq \min\{rank(A), rank(B)\} \ .$$

Proof. Apply Lemma 2 to $A \cdot B$ and its transpose.

4.1 Triangle inequalities

Theorem 8. *Triangle inequalities define facets of R_f and of $R_f(r^1, r^2, r^3)$ where r^1, r^2 and r^3 are the corner rays of the maximal lattice-free triangle.*

Proof. Since $k = 3$ and B_ψ is a triangle, both $\bar{M}$ and $\bar{X}$ have rank 3. By Corollary 2, S has rank 3 too. It implies that the columns of S are affinely independent. Since they all satisfy with equality the inequality $\sum_{i=1}^{3} \psi(r^i)s_i \geq 1$, this inequality defines a facet of $R_f(r^1, r^2, r^3)$. By Theorem 5, ψ defines a facet of R_f.

4.2 Quadrilateral inequalities

When $k = 4$, both $\bar{M}$ and $\bar{X}$ have rank 3. By Lemma 2, we have

$$3 = rank(\bar{M}) = rank(\bar{X} \cdot S) = rank(S) - dim(\mathcal{N}(\bar{X}) \cap \mathcal{C}(S)) .$$

Since $rank(\bar{X}) = 3$, we have that $\mathcal{N}(\bar{X})$ is a one-dimensional linear space. Hence $dim(\mathcal{N}(\bar{X}) \cap \mathcal{C}(S)) \leq 1$ and $rank(S) = 4$ if and only if $\mathcal{N}(\bar{X}) \subseteq \mathcal{C}(S)$.

Theorem 9. *Consider a maximal lattice-free quadrilateral with vertices x^i, integral point y^i on edge $x^i x^{i+1}$ (indices taken modulo 4) and corner rays r^i, $i = 1, \ldots, 4$. The corresponding quadrilateral inequality defines a facet of $R_f(r^1, r^2, r^3, r^4)$ (and therefore of R_f) if and only if there is no $t \in \mathbb{R}_+$ such that the point y^i divides the edge joining x^i to x^{i+1} in a ratio t for odd i and in a ratio $1/t$ for even i, i.e.*

$$\frac{||y^i - x^i||}{||y^i - x^{i+1}||} = \begin{cases} t & \text{for } i = 1, 3 \\ \frac{1}{t} & \text{for } i = 2, 4 \end{cases} .$$

Proof. Let F be the face of $R_f(r^1, \ldots, r^4)$ defined by $\sum_{i=1}^{4} \psi(r^i)s_i = 1$. As $f + r^i = x^i$ is on the boundary of B_ψ, we have $\psi_i(r^i) = 1$ for $i = 1, \ldots, 4$. Hence, if $s \in F$ then $\sum_{i=1}^{4} s_i = 1$. Recall that $R_f(r^1, \ldots, r^k)$ is the convex hull of vectors in the set $H := \{s \in \mathbb{R}^4_+ \mid f + \sum_{i=1}^{4} r^i s_i \text{ is integral}\}$. Thus, if F is a facet, then there exists four affinely independent vectors s^j, for $j = 1, \ldots, 4$, in H with

$$\sum_{i=1}^{4} s_i^j = 1 \qquad \text{and} \qquad z^j = f + \sum_{i=1}^{4} r_i s_i^j = \sum_{i=1}^{4} (f + r_i) s_i^j \quad \text{integer} .$$

This implies that z^j is in the convex hull of $x^1, \ldots, x^4$, for $j = 1, \ldots, 4$. Theorem 2 shows that the only integral points in B_ψ are the points $y^1, \ldots, y^4$. Moreover, for each $j = 1, \ldots, 4$, there is a unique convex combination of $x^1, \ldots, x^4$ that produces y^j, namely column j of matrix S. In other words, F is a facet if and only if the columns of S are affinely independent. Observe that the columns of S are affinely independent if and only if they are linearly independent since the sum of the entries in any column of S is 1. It follows that F is a facet if and only if $rank(S) = 4$.

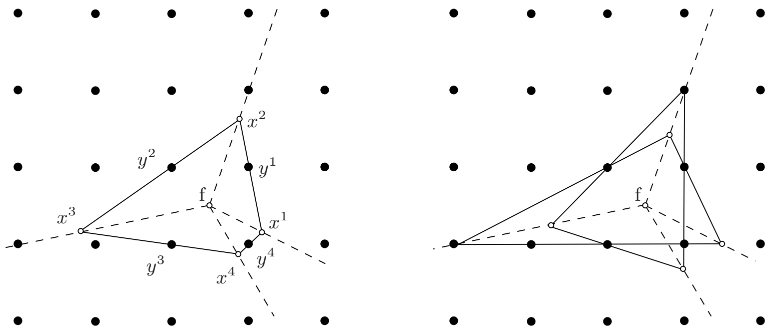
Let $u = (1, -1, 1, -1)^T$. By Theorem 2 iv), the points $y^1, \dots, y^4$ are the vertices of a parallelogram. This implies that $\bar{M} \cdot u = 0$. Then (6) gives $\bar{X} \cdot S \cdot u = 0$. We now have two cases:

i) $S \cdot u = 0$. Then $rank(S) \leq 3$ and Corollary 3 shows that $rank(S) = 3$. Solving the linear system $S \cdot u = 0$ gives $\alpha = 1 - \beta = \gamma = 1 - \delta$. This is equivalent to the ratio condition of the statement.

ii) $S \cdot u \neq 0$. Then for $v = S \cdot u$ we have $\bar{X} \cdot v = 0$, and as $v \neq 0$, we have that $\mathcal{N}(\bar{X})$ is the linear space spanned by v. Since v is obtained as a linear combination of the columns of S, we have $\mathcal{N}(\bar{X}) \subseteq \mathcal{C}(S)$ and by Lemma 2 we get $rank(S) = 4$. Since all the columns of S satisfy with equality the inequality $\sum_{i=1}^{4} \psi(r^i) s_i \geq 1$, this inequality defines a facet of $R_f(r^1, r^2, r^3, r^4)$. By Theorem 5, ψ defines a facet of R_f.

We illustrate the condition in Theorem 9 by a couple of examples. This condition implies that the quadrilateral inequality generated from the square whose edges contain the integral points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in their middle (usually called *octahedron* inequality) does not define a facet. However, if one tilts just one edge of the square around its (integral) middle point, the resulting trapezoid has three distinct ratios $\frac{||y^i - x^i||}{||y^i - x^{i+1}||}$. Therefore Theorem 9 states that the resulting quadrilateral inequality defines a facet of R_f.

We give another more complicated example, see Figure 6. Let $f = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ and Q the quadrilateral with vertices $x^1 = \begin{pmatrix} \frac{7}{6} \\ \frac{1}{6} \end{pmatrix}$, $x^2 = \begin{pmatrix} \frac{7}{8} \\ \frac{13}{8} \end{pmatrix}$, $x^3 = \begin{pmatrix} -\frac{7}{6} \\ \frac{1}{6} \end{pmatrix}$, $x^4 = \begin{pmatrix} \frac{7}{8} \\ -\frac{1}{8} \end{pmatrix}$.



(a) The quadrilateral Q. (b) The two triangles T_1 and T_2.

Fig. 6: Illustration for the second example.

Edge x^1x^2 contains integral point $y^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with ratio $\frac{||y^1-x^1||}{||y^1-x^2||} = \frac{4}{3}$.

Edge x^2x^3 contains integral point $y^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with ratio $\frac{||y^2-x^2||}{||y^2-x^3||} = \frac{3}{4}$.

Edge x^3x^4 contains integral point $y^3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with ratio $\frac{||y^3-x^3||}{||y^3-x^4||} = \frac{4}{3}$.

Edge x^4x^1 contains integral point $y^4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with ratio $\frac{||y^4-x^4||}{||y^4-x^1||} = \frac{3}{4}$.

Theorem 9 states that the quadrilateral inequality obtained from Q is not a facet.

Indeed, it can be obtained as a convex combination of two triangle inequalities, each with a multiplier $\frac{1}{2}$. The first triangle T_1 has vertices $\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{4}{5} \\ \frac{7}{5} \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$. The second triangle has vertices $\begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -\frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$. Both triangles have all four points y^1, y^2, y^3, y^4 on their boundaries. The corner rays of Q are $r^1 = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$, $r^2 = \begin{pmatrix} \frac{3}{8} \\ \frac{9}{8} \end{pmatrix}$, $r^3 = \begin{pmatrix} -\frac{5}{3} \\ -\frac{1}{3} \end{pmatrix}$, $r^4 = \begin{pmatrix} \frac{3}{8} \\ -\frac{5}{8} \end{pmatrix}$. Triangle T_1 has corner rays positive multiples of r^1, r^2 and r^3. Triangle T_2 has corner rays positive multiples of r^2, r^3 and r^4. If ψ, ψ_1 and ψ_2 denote the functions defined by Q, T_1 and T_2 respectively, it is easy to verify that $\psi = \frac{1}{2}\psi_1 + \frac{1}{2}\psi_2$ in each of the cones r^ir^{i+1} (indices defined modulo 4). Indeed, each of these functions is linear in each of the cones. So it is sufficient to verify the equality $\psi(r) = \frac{1}{2}\psi_1(r) + \frac{1}{2}\psi_2(r)$ in each of the directions r^i, $i = 1, \ldots, 4$. In direction r^1 we have $\psi_1 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} = 1$ and $\psi_2 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{pmatrix} = 1$. This implies $\psi_1(r^1) = \frac{2}{3}$ and $\psi_2(r^1) = \frac{4}{3}$. Therefore $\psi(r^1) = \frac{1}{2}\psi_1(r^1) + \frac{1}{2}\psi_2(r^1)$ as required. Similarly, for the other rays, we find $\psi_1(r^2) = \frac{5}{4}$ and $\psi_2(r^2) = \frac{3}{4}$; $\psi_1(r^3) = \frac{4}{3}$ and $\psi_2(r^3) = \frac{2}{3}$; $\psi_1(r^4) = \frac{3}{4}$ and $\psi_2(r^4) = \frac{5}{4}$.

5 Degenerate Triangle and Quadrilateral Inequalities

Theorem 10. *A vertex-degenerate triangle inequality defines a facet of* R_f *if and only if the edge of* T *opposite* f *contains at least two integral points in its interior.*

Theorem 11. *An edge-degenerate triangle inequality defines a facet of* R_f *if and only if at least one of the two edges not containing* f *contains two integral points* y *with* $\psi(y - f) = 1$.

Theorem 12. *Vertex-degenerate and edge-degenerate quadrilateral inequalities never define facets of* R_f.

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