Generalized Intersection Cuts and a new cut generating paradigm

Egon Balas[*†] and François Margot[†]
Carnegie Mellon University

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Abstract

Intersection cuts are generated from a polyhedral cone and a convex set S whose interior contains no feasible integer point. We generalize these cuts by replacing the cone with a more general polyhedron C. The resulting generalized intersection cuts dominate the original ones. This leads to a new cutting plane paradigm under which one generates and stores the intersection points of the extreme rays of C with the boundary of S rather than the cuts themselves. These intersection points can then be used to generate in a non-recursive fashion cuts that would require several recursive applications of some standard cut generating routine. A procedure is also given for strengthening the coefficients of the integer-constrained variables of a generalized intersection cut. The new cutting plane paradigm yields a new characterization of the closure of intersection cuts and their strengthened variants. This characterization is minimal in the sense that every one of the inequalities it uses defines a facet of the closure.

1 Introduction

Consider the mixed integer program

$$\min\{cx : Ax \geq b,\ x \geq 0,\ x_j \in \mathbb{Z},\ j \in N_1 \subseteq N\} \qquad \text{(MIP)}$$

where A is $m \times n$ and $N := \{1, \ldots, n\}$. Let $P := \{x \in \mathbb{R}^n : Ax \geq b,\ x \geq 0\}$, and $P_I := \{x \in P : x_j \in \mathbb{Z},\ j \in N_1\}$.

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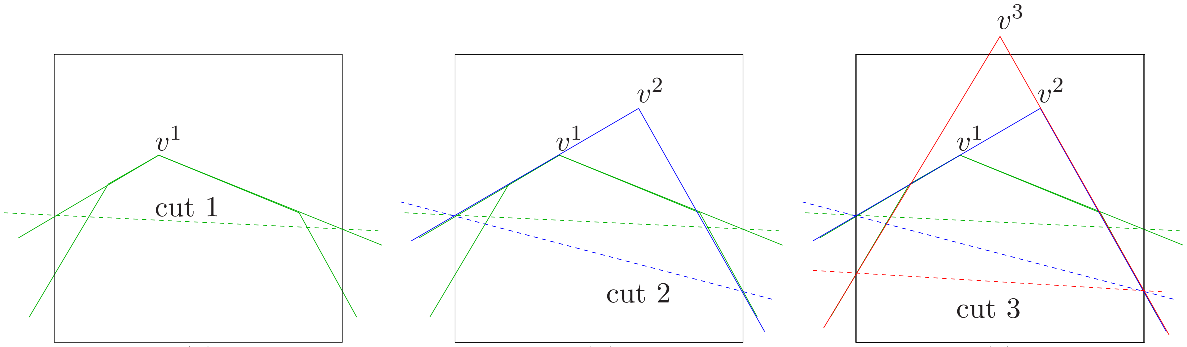
Intersection cuts [2] are derived from a displaced polyhedral cone formed by the intersection of n halfspaces from among those defining P. Such a cone is called simplicial. The intersection of the corresponding hyperplanes is the apex of the cone. To be specific, if $C(v)$ is the displaced simplicial cone with apex at v, an intersection cut can be derived by intersecting the extreme rays of $C(v)$ with the boundary of any convex set S whose interior contains v but no feasible integer point. So the crucial requirement on S is that $P_I \cap \operatorname{int} S = \emptyset$. A convex set S satisfying this requirement will be called *proper*. The intersection points of the n extreme rays of $C(v)$ with the boundary of S define a hyperplane $\alpha x = \beta$ such that the inequality $\alpha x \geq \beta$ cuts off v but no feasible integer point. If an extreme ray is contained in its entirety in S, we say that it intersects the boundary of S at infinity.

Originally the cone used for generating intersection cuts was the one whose apex is $\bar{x}$, the minimizer of cx over P. However, as the recent procedure of generating lift-and-project cuts [3] from the LP tableau [6, 4, 15] has demonstrated, stronger intersection cuts can be obtained from a cone whose apex is a point v corresponding to an *infeasible* basic solution, such that the cone with apex at v contains $\bar{x}$, the vertex to be cut off. This is illustrated in Figure 1, where (a) pictures an intersection cut derived from a vertex $\bar{x} = v^1$ of P, while (b) and (c) show cuts derived from points v^2 and v^3 corresponding to infeasible basic solutions. Such points can be viewed as vertices (0-dimensional faces) of the hyperplane arrangement $\mathcal{A}(P)$ in $\mathbb{R}^n$ associated with P, which is the collection of hyperplanes defining the constraints of P (see, for instance, chapter 18 of [12]). A vertex of $\mathcal{A}(P)$ is the intersection point of n hyperplanes of $\mathcal{A}(P)$, i.e. there is a one to one correspondence between the vertices of $\mathcal{A}(P)$ and the basic solutions (feasible or not) of the system defining P.

Lately, the intersection cut framework has been found useful in the study of valid inequalities derived from multiple rows of the simplex tableau (see, for instance, [1, 7, 9]).



(a) (b) (c)

Figure 1:

Our purpose here is to investigate the possibility of deriving intersection-type cuts from polyhedra other than simplicial cones, whose extended edges or extreme rays intersect bd S at points beyond the hyperplane $\alpha x = \beta$, thereby providing cuts that dominate $\alpha x \geq \beta$.

If we view the Balas-Perregaard procedure of [6] as a generalization of the original intersection cuts derived from vertices of P to intersection cuts derived from vertices of $\mathcal{A}(P)$, then the approach of this paper can be viewed as a further generalization along those lines.

Our paper is structured as follows. Section 2 introduces the generalized intersection cuts as valid inequalities obtained by intersecting the extreme rays of some relaxation of P with the boundary of a proper convex set S. The main result of this section, and of the paper, Theorem 4, shows how to obtain valid cuts from a collection of such intersection points and gives conditions under which these cuts dominate the standard intersection cuts. The next two sections, 3 and 4, describe the procedure for generating the intersection points. Section 5 states the new cut generating paradigm suggested by the results of section 2. The next section, 6, gives a geometric interpretation of intersection cuts with strengthened coefficients corresponding to integer-constrained variables, which makes it possible to strengthen in a similar fashion the coefficients of generalized intersection cuts. Finally, section 7 discusses the closure of P under the cut generating procedure discussed here, and its relationship to the closure of P under other cut generating procedures.

2 Intersection-type cuts from non-conic polyhedra

Let C_1 be a displaced simplicial cone containing $\bar{x}$, whose n facets are facets of P and whose apex v^1 is a vertex of the hyperplane arrangement associated with P. Further, let S be a convex set containing v^1 and such that $P_I \cap \operatorname{int} S = \emptyset$, and let $\alpha^1 x \geq \beta_1$ be the intersection cut derived from C_1 and S. Consider now the polyhedron $C := C_1 \cap H^+$, where H^+ is a facet-defining halfspace for P, such that $v^1 \in H^+$ and H, the hyperplane bounding H^+, is intersected by some of the extreme rays of C_1 at a point $v \in \operatorname{int} S$. C inherits from C_1 its vertex v^1 and possibly some of its extreme rays, but intersecting C_1 with H^+ creates new vertices $v \in \operatorname{int} S$ (unless $v^1 \in H$) and new edges (possibly infinite) of C originating at the new vertices. (See figure 2(a) where H intersects a single extreme ray of C_1, and 2(b), where it intersects two.) In the case where $v^1 \in H$, no new vertices are created; only new edges. Actually, intersecting C_1 with H^+ may also create vertices of C outside S, but these are not of interest to us, since they play no role in generating cuts from S. Therefore the edges of C joining a vertex in S to a vertex outside S will be treated the same way as the (and therefore called) infinite edges (extreme rays) of C emanating from a vertex of C belonging to $\operatorname{int} S$.

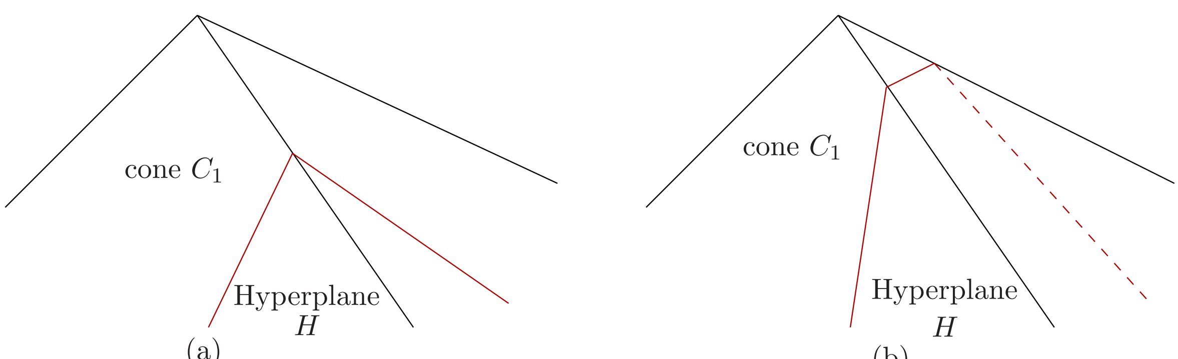


Figure 2:

Assume first that $v^1 \in \operatorname{int} H^+$.

Theorem 1. *If k extreme rays of C_1, $1 \leq k \leq n-1$, intersect H at a point $v \in \operatorname{int} S$, then C has $(k+1)(n-k)$ edges intersecting $\operatorname{bd} S$ (possibly at infinity).*

Proof. Suppose k extreme rays r_j of C_1 intersect H at a point $v \in \operatorname{int} S$. This creates k new vertices of C contained in $\operatorname{int} S$. Each new vertex (intersection point $r_j \cap H$) is incident with $n-1$ new edges of C, each consisting of the intersection of H with $n-2$ of the $n-1$ facets of C_1 whose intersection is r_j. Of these $n-1$ new edges, $k-1$ join the new vertex $r_j \cap H$ to each of the remaining $k-1$ new vertices, while the remaining $n-k$ new edges have only one end in $\operatorname{int} S$. This is a total of $k(n-k)$ new edges, some (or all) of them infinite. Further, the $n-k$ extreme rays of C_1 that do not intersect H in the interior of S, are of course also edges (extreme rays) of C. This gives a total of $k(n-k)+(n-k) = (k+1)(n-k)$ edges of C that intersect $\operatorname{bd} S$, possibly at infinity. $\square$

Note that if $k = n$, then $(C_1 \cap H^+) \subseteq P \subset \operatorname{int} S$, hence $P_I = \emptyset$.

If $v^1 \in H$, which implies that the solution associated with v^1 is degenerate, then the "new" vertices $r_j \cap H$ coincide with v^1. The number of new rays created remains the same as in the theorem. For details of this degenerate case, see section 3.

We need a definition of dominance relation between valid cuts, i.e. inequalities valid for P_I but not for P, that have coefficients of arbitrary sign. Let $\alpha^1 x \geq \beta_1$ and $\alpha^2 x \geq \beta_2$ be two such inequalities called 1 and 2, respectively.

Definition. *Inequality 2 dominates 1 on P if for every $x \in P$, $\alpha^2 x \geq \beta_2$ implies $\alpha^1 x \geq \beta_1$. Inequality 2 strictly dominates 1 if, in addition, there exists $x \in P$ such that $\alpha^1 x \geq \beta_1$ but $\alpha^2 x < \beta_2$.*

The next proposition gives necessary and sufficient conditions for domination that are easier to check than those of the definition.

For $i = 1, 2$, denote $D_i := \{x \in P : \alpha^i x = \beta_i\}$.

Proposition 2. *Suppose there exists* $v \in P$ *such that* $\alpha^i v < \beta_i$, $i = 1, 2$. *Inequality 2 dominates 1 on* P *if and only if* $\alpha^1 x \geq \beta_1$, *for all* $x \in D_2$. *The domination is strict if and only if (a)* $D_2 \subsetneqq D_1$, *or (b)* $D_2 \neq \emptyset$ *and* $\alpha^1 x > \beta_1$, *for some* $x \in D_2$.

Proof. If inequality 2 does not dominate 1 on P, there exists $x \in P$ with $\alpha^1 x < \beta_1$ and $\alpha^2 x \geq \beta_2$. Since there exists $v \in P$ such that $\alpha^i v < \beta_i$ for $i = 1, 2$, there exists some point $z \in P$ on the segment (v, x) such that $z \in D_2$ and $\alpha^1 z < \beta_1$. Conversely, if there exists $x \in D_2$ with $\alpha^1 x < \beta_1$, then x satisfies inequality 2 but violates 1, hence 2 does not dominate 1 on P.

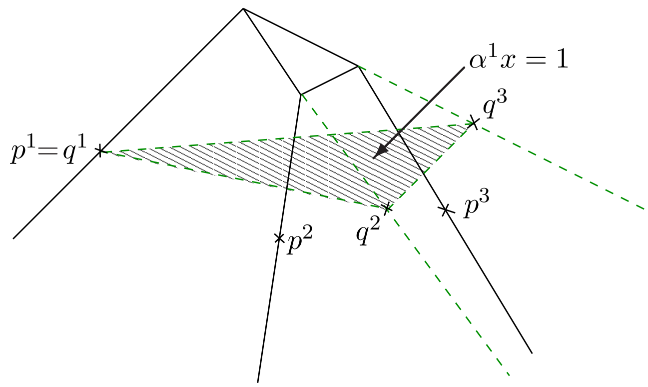
Now assume inequality 2 dominates 1.

If $D_2 \subsetneqq D_1$, let $x \in D_1 \setminus D_2$. If $\alpha^2 x \geq \beta_2$ then there exists $x^1 \in P$ on the segment (v, x) with $\alpha^2 x^1 = \beta_2$ and $\alpha^1 x^1 < \beta_1$, a contradiction. Thus we have $\alpha^2 x < \beta_2$ and the domination is strict. If $D_2 \neq \emptyset$ and $\alpha^1 x > \beta_1$ for some $x \in D_2$, then there exists $x^1 \in P$ on the segment (v, x) with $\alpha^1 x^1 = \beta_1$ and $\alpha^2 x^1 < \beta_2$ and the domination is strict. Conversely, if D_2 is not strictly contained in D_1, $D_2 \neq \emptyset$ and $\alpha^1 x = \beta_1$ for all $x \in D_2$, then $D_2 = D_1$ and inequality 2 does not strictly dominate 1. □

From now on, for all inequalities $\alpha x \geq \beta$ we assume w.l.o.g. that $\beta \in \{1, 0, -1\}$.

Next we define a class of valid inequalities derived by intersecting extreme rays of C with $\operatorname{bd} S$, whose members dominate the inequality $\alpha^1 x \geq \beta_1$, and some of whose members strictly dominate the latter. First we note that the number of inequalities in our class that strictly dominate $\alpha^1 x \geq \beta_1$ depends on how the convex set S is chosen. Actually, for certain sets S such inequalities do not exist: this is the case, for instance, if $\operatorname{bd} S$ contains the entire set $C_1 \cap \{x : \alpha^1 x = \beta_1\}$. Thus, if $S = \{x \in \mathbb{R}^n : \pi_0 \leq \pi x \leq \pi_0 + 1\}$, and every extreme ray of C_1 intersects the hyperplane $\pi x = \pi_0$, then there exists no inequality that strictly dominates $\alpha^1 x \geq \beta_1$.

Since maximal convex sets whose interior contains no integer point are polyhedral (see [14]), we will assume that S is a polyhedron. Now consider the $(n-1)$-dimensional simplex $C_1 \cap \{x : \alpha^1 x = \beta_1\}$ whose n vertices q^j, $j = 1, \ldots, n$, are the intersection points of the extreme rays of C_1 with $\operatorname{bd} S$. We will assume that $C_1 \cap \{x : \alpha^1 x = \beta_1\} \not\subseteq \operatorname{bd} S$, or, equivalently, that the n intersection points q^j do not lie all on the same face of S.



q^1, q^2, q^3 – intersection points of extreme rays of C_1 with $\operatorname{bd} S$

p^1, p^2, p^3 – intersection points of extreme rays of C with $\operatorname{bd} S$

Figure 3:

In order to simplify the discussion to follow, we also assume that S is bounded. This of course is not the case for some of the most important cut generating sets S, but it is easily handled by assuming that any ray that intersects $\operatorname{bd} S$ at infinity, gives rise to a coefficient ϵ rather than 0, where ϵ is orders of magnitude smaller than the coefficients corresponding to intersection points at a finite distance. This will be discussed in detail at the end of this section.

Let r_j, $j \in Q$, be the extreme rays (i.e., using our terminology, edges with only one end in $\operatorname{int} S$) of C, and let $p^j := r_j \cap \operatorname{bd} S$, $j \in Q$ be the intersection points of r_j with the boundary of S. Further, let $Q_1 := \{j \in Q : p^j = q^j\}$, i.e. Q_1 is the set of indices j such that r_j is an extreme ray of both C_1 and C, and let $Q_2 := Q \setminus Q_1$. Finally, let k be, as in Theorem 1, the number of extreme rays of C_1 that intersect H (strictly) before intersecting $\operatorname{bd} S$. Within the set of intersection points indexed by Q_2, it is useful to distinguish between those p^j that are convex combinations of the points q^ℓ and those that are not; let their respective index sets be Q_2' and Q_2''. The points p^j,

$j \in Q_2'$, being convex combinations of q^ℓ, lie on the hyperplane $\alpha^1 x = \beta_1$; whereas the points p^j, $j \in Q_2''$ do not, and as the next theorem will show, they satisfy $\alpha^1 p^j > \beta_1$.

The intersection points q^j, p^j and the sets Q_1, Q_2 are illustrated in Figure 3, where the ray that is not cut by the hyperplane H intersects $\operatorname{bd} S$ in the point $p^1 = q^1$, whereas the two new rays, contained in H, intersect $\operatorname{bd} S$ in the points p^2 and p^3. Thus $Q_1 = \{1\}$ and $Q_2 = \{2, 3\}$. Further, $Q_2' = \emptyset$, $Q_2'' = \{2, 3\}$, and both p^2 and p^3 lie beyond the hyperplane through q^1, q^2 and q^3 that defines the intersection cut from C_1.

Theorem 3. *Every intersection point $p^j = r_j \cap \operatorname{bd} S$, $j \in Q$, satisfies $\alpha^1 p^j \geq \beta_1$. Every intersection point p^j, $j \in Q_2''$, satisfies $\alpha^1 p^j > \beta_1$.*

Proof. Notice that $C_1 \cap \{x : \alpha^1 x \leq \beta_1\}$ is an n-dimensional simplex contained in S. From $C \subset C_1$, it follows that $C \cap \{x : \alpha^1 x \leq \beta_1\}$ is bounded and contained in S. Hence any edge of C incident with a vertex in $C \cap \{x : \alpha^1 x < \beta_1\}$ intersects the hyperplane $\alpha^1 x = \beta_1$ before, or concomitantly with, the boundary of S. Thus p^j, the intersection of r_j with $\operatorname{bd} S$, satisfies $\alpha^1 p^j \geq \beta_1$.

Now $\alpha^1 p^j > \beta_1$ if and only if r_j intersects $\{x : \alpha^1 x = \beta_1\}$ strictly before intersecting $\operatorname{bd} S$. Since the $n - k$ extreme rays r_j of C that are also extreme rays of C_1 intersect $\operatorname{bd} S$ concomitantly with $\{x : \alpha^1 x = \beta_1\}$, their intersection points with $\operatorname{bd} S$ obviously satisfy $\alpha^1 p^j = \beta_1$. The intersection points p^j of the remaining $k(n-k)$ extreme rays (edges) of C belong to two classes. Those indexed by Q_2', being convex combinations of the q^ℓ, $\ell = 1, \ldots, 2$, clearly satisfy $\alpha^1 x = \beta_1$. Those indexed by Q_2'', not being such a combination, do not satisfy $\alpha^1 p^j = \beta_1$. From the first paragraph of the proof, they do not satisfy $\alpha^1 p^j < \beta^1$ either. Hence they satisfy $\alpha^1 p^j > \beta^1$, i.e. their generating rays intersect the hyperplane $\{x : \alpha^1 x = \beta_1\}$ strictly before intersecting $\operatorname{bd} S$. □

We now state the central result of this paper.

As before, let v^1 be the apex of the initial cone C_1, let $\alpha^1 x \geq \beta_1$ be the (standard) intersection cut from C_1 and S, and for $j = 1, \ldots, n$, let q^j be the intersection point of the j-th extreme ray of C_1 with $\operatorname{bd} S$.

Theorem 4. *Consider the three systems*

$$\alpha p^j \geq \bar{\beta}, \quad j \in Q \tag{$1)_{\bar{\beta}}$$

for $\bar{\beta} = 1$, $\bar{\beta} = -1$ and $\bar{\beta} = 0$.

A basic feasible solution $\bar{\alpha}$ to any of the three systems $(1)_{\bar{\beta}}$, such that $\bar{\alpha}v^1 < \bar{\beta}$, yields a valid cut $\bar{\alpha}x \geq \bar{\beta}$ for P_I. If, in addition, $\bar{\alpha}$ satisfies

$$\alpha q^j \leq \bar{\beta} \text{ for } j = 1, \ldots, n, \tag{2}$$

where $\bar{\beta}$ has the value that yields $\bar{\alpha}$, then $\bar{\alpha}x \geq \bar{\beta}$ dominates $\alpha^1 x \geq \beta_1$ on C, with strict domination if $\alpha q^j < \bar{\beta}$ for at least one j.

Proof. (a) The inequality $\bar{\alpha}x \geq \bar{\beta}$ does not cut off any point of P_I, since the region it cuts off, $C \cap \{x : \bar{\alpha}x < \bar{\beta}\}$, contains no such point. To see this, note that the righthand side of $\bar{\alpha}x \geq \bar{\beta}$ is determined by the condition $\bar{\alpha}v^1 < \bar{\beta}$. Next, $P_I \cap \operatorname{int} S = \emptyset$ by the definition of S, hence if any point of P_I is cut off, it must be contained in the set $F := C \cap \{x : \bar{\alpha}x < \bar{\beta}\} \cap \operatorname{bd} S$. We claim that F is empty. For if not, then F contains an intersection point, say p^h, of $\operatorname{bd} S$ with an edge of C. But each such intersection point is associated with an inequality of the system $(1)_{\bar{\beta}}$, so if p^h is cut off, i.e. $\bar{\alpha}p^h < \bar{\beta}$, then $\bar{\alpha}$ violates the inequality of $(1)_{\bar{\beta}}$ corresponding to p^h.

(b) If $\bar{\alpha}x \geq \bar{\beta}$ does not dominate $\alpha^1 x \geq \beta_1$ on C, then there exists $y \in C$ such that $\bar{\alpha}y \geq \bar{\beta}$ but $\alpha^1 y < \beta_1$. Since $y \in C_1 \cap \{x : \alpha^1 x < \beta_1\}$, y is a convex combination of v^1 and the n points q^j, $j = j_1, \ldots, j_n$. Let w be the point on the hyperplane $\alpha^1 x = \beta^1$ such that y lies on the segment (v_1, w). Then $\alpha^1 w = \beta_1$ and w is the convex combination of the point q^j, $j = j_1, \ldots, j_n$. Also, $\bar{\alpha}w > \bar{\beta}$, since the segment (y, w) has positive length. But then $\bar{\alpha}q^{j^*} > \bar{\beta}$ for some $j^* \in j_1, \ldots, j_n$, i.e. condition (2) of the Theorem is violated. This proves that if condition (2) holds, then $\bar{\alpha}x \geq \bar{\beta}$ dominates $\alpha^1 x \geq \beta_1$ on C. □

The above Theorem gives an easily verifiable sufficient condition for the domination of $\alpha^1 x \geq \beta_1$ by an inequality defined by a basic solution of one of the systems $(1)_{\bar{\beta}}$. A tighter, necessary and sufficient condition is given in the next Theorem.

Theorem 5. *Let $T := C \cap \{x \in \mathbb{R}^n : \alpha^1 x = \beta_1\}$. The inequality $\bar{\alpha}x \geq \bar{\beta}$ dominates $\alpha^1 x \geq \beta_1$ on C if and only if $\bar{\alpha}t \leq \bar{\beta}$ for all $t \in T$. The domination is strict if $\bar{\alpha}t < \bar{\beta}$ for at least one $t \in T$.*

Proof. Let $\bar{T} = C \cap \{x \in \mathbb{R}^n : \bar{\alpha}x = \bar{\beta}\}$ and $P_{\bar{\alpha}} = \{p^j : \bar{\alpha}p^j = \bar{\beta}\}$. Observe that $\bar{T}$ is the convex hull of the points $P_{\bar{\alpha}}$ and, since at least n affinely independent points are in $P_{\bar{\alpha}}$, $dim(\bar{T}) = n - 1$. Moreover, any $p \in P_{\bar{\alpha}}$ satisfies $\alpha^1 p \geq \beta_1$, as all points $x \in C$ with $\alpha^1 x < \beta_1$ are in $int(S)$ and p is in $\text{bd}\, S$.

We claim that $\bar{\alpha}t < \bar{\beta}$ for some $t \in T$ if and only if $\alpha^1 z > \beta_1$ for some $z \in \bar{T}$. Indeed, if such a t exists, then as $dim(T) \leq dim(\bar{T}) = n - 1$, we have that $\bar{T}$ is not contained in T and there exists some $z \in P_{\bar{\alpha}}$ proving the claim. Conversely, if such a z exists, then taking t as the intersection of the segment zv^1 with $\{x \in \mathbb{R}^n : \alpha^1 x = \beta_1\}$ verifies the claim. The result now follows from Proposition 2. □

It should be mentioned at this point that dominance relations between split cuts have also been studied in [10].

Theorem 4 can be used to generate valid cuts stronger than the intersection cut $\alpha^1 x \geq \beta_1$. Any basic feasible solution $\bar{\alpha}$ to any of the three systems $(1)_{\bar{\beta}}$ yields a valid cut, and if, in addition, $\bar{\alpha}$ satisfies the condition (2), then this cut dominates $\alpha^1 x \geq \beta_1$. If one wants to generate only inequalities that dominate $\alpha^1 x \geq \beta_1$, then condition (2) should be added to whichever of the three systems $(1)_{\bar{\beta}}$ is being used. However, cuts that do not dominate $\alpha^1 x \geq \beta_1$ may be very strong in certain directions and thus may be desirable.

Theorem 5, while giving a necessary and sufficient condition for a generalized intersection cut from a polyhedron C to dominate an intersection cut $\alpha^1 x \geq \beta_1$ from a given cone C_1, also gives the condition for the dominance to be strict. This leaves open the question, whether there exist any situations in which none of the valid cuts obtainable from C strictly dominate $\alpha^1 x \geq \beta_1$? The next Theorem addresses this question.

Let us again denote $q^j := r_j \cap \mathrm{bd}\, S$, where r_j is the j-th extreme ray of the cone C_1 with apex at vertex v^1, $j \in N := \{1, \ldots, n\}$, and let H be one of the hyperplanes activated in generating C, with $v^1 \in H^+$.

Theorem 6. *Let $N^{\mathrm{cut}} \subset N$ be the index set of $q^j \notin H^+$. Let E be the set of edges e of the simplex induced by q^j, $j \in N$, such that $e \in \mathrm{bd}\, S$ and $\mathrm{int}\,(e) \cap H$ is nonempty. If for each $j \in N^{\mathrm{cut}}$ there exists $q^{k(j)} \in N \setminus \{j\}$ with $e = (q^j, q^{k(j)}) \in E$, then none of the valid cuts obtained from C strictly dominates $\alpha^1 x \geq \beta_1$.*

Proof. Suppose $\bar{\alpha} x \geq \beta_1$ strictly dominates $\alpha^1 x \geq \beta_1$. Then there exists $j \in N$ with $\bar{\alpha} q^j < \alpha^1 q^j = \beta_1$. This implies that $j \in N^{\mathrm{cut}}$ and thus there exists $e = (q^j, q^{k(j)}) \in E$ for some $k(j)$. Since $\mathrm{int}\,(e) \cap H$ is nonempty, there exists a ray of C whose intersection z with $\mathrm{bd}\, S$ is in $\mathrm{int}\,(e)$. From the definition of $\bar{\alpha}$, we have $\bar{\alpha} z \geq \beta_1$. Since $\bar{\alpha} q^j < \beta_1$, we must have $\bar{\alpha} q^{k(j)} > \beta_1 = \alpha^1 q^{k(j)}$, contrary to the assumption that $\bar{\alpha} x \geq \beta_1$ strictly dominates $\alpha^1 x \geq \beta_1$. □

As long as the boundary of S does not contain the entire simplex $C_1 \cap \{x : \alpha^1 x = \beta_1\}$, intersecting C_1 with H will always create some new intersection points $p^\ell = r_\ell \cap \mathrm{bd}\, S$ such that $\alpha^1 p^\ell > \beta_1$, which can be used, together with other points indexed by Q of $(1)_{\bar{\beta}}$, to generate a cut $\bar{\alpha} x \geq \beta_1$ that strictly dominates $\alpha^1 x \geq \beta_1$. What Theorem 6 says is that if its condition holds, then the cut $\bar{\alpha} x \geq \beta_1$ is not valid, since it cuts off at least one of the intersection points indexed by Q, say q^1. Of course, if the procedure of activating facets of P is iterated as it will be outlined later on, then such an intersection point q^1 may be cut off by a subsequently activated hyperplane, i.e. q^1 may be removed from the set $(1)_{\bar{\beta}}$ indexed by Q, in which case the cut $\bar{\alpha} x \geq \beta_1$ which dominates $\alpha^1 x \geq \beta_1$ may become valid.

In deriving the above results, in order to simplify our proofs, we assumed that the proper convex set S is bounded. We now take a look at what it implies for our main result, Theorem 4, if we allow S to be unbounded. Suppose the extreme ray $r_j := \{x : x = \tilde{a}_0 - \tilde{a}_j s_j,\ s_j \geq 0\}$ of C is contained in S, i.e. $r_j \cap \operatorname{bd} S = \emptyset$. This situation can be represented by having the intersection occur for an infinitely large s_j, i.e. having an intersection point of the form $(-\tilde{a}_j)L$, where $L > 0$ is sufficiently large to render $\tilde{a}_0$ negligible. Then clearly $\alpha \in \mathbb{R}^n$ satisfies $\alpha(-\tilde{a}_j)L \geq 1$ if and only if it satisfies $\alpha(-\tilde{a}_j) \geq \varepsilon$ for arbitrarily small $\varepsilon > 0$. It follows that Theorem 4 remains valid for proper convex sets S that are unbounded, if for every intersection point p^j, $j \in Q$, "at infinity," i.e. of the form $p^j = (-\tilde{a}_j)L$, we replace the inequality $\alpha p^j \geq 1$ of the system $(1)_{\bar{\beta}}$ with $\alpha \bar{p}^j \geq 0$, where $\bar{p}^j := -\tilde{a}_j$. To see that this apparent weakening of the system $(1)_{\bar{\beta}}$ cannot result in a solution α that yields an invalid inequality, note that such an inequality would have to cut off some integer point $x^* \notin \operatorname{int} S$ of C. But by the convexity of C, any such point can be expressed as $x^* = \sum_{j\in Q'} p^j \lambda_j + \sum_{\ell \in Q''} \bar{p}^\ell \mu_\ell$, with $\sum_{j \in Q'} \lambda_j = 1$, $\lambda_j \geq 0$, $j \in Q'$, $\mu_\ell > 0$, $\ell \in Q''$ where Q' and Q'' index the nonhomogeneous and homogeneous inequalities, respectively, of $(1)_{\bar{\beta}}$. It follows that $\alpha x^* \geq \bar{\beta}$ for all α satisfying $\alpha p^j \geq \bar{\beta}$, $j \in Q'$ and $\alpha \bar{p}^\ell \geq 0$, $\ell \in Q''$.

3 Generating the rays of C and their intersection points with $\operatorname{bd} S$

Next we discuss a practical way of generating the extreme rays of C and their intersection points with $\operatorname{bd} S$.

Suppose that after introducing m surplus variables into the system $Ax \geq b$, the cone C_1 is associated with the simplex tableau in $m + n$ variables

$$\begin{aligned} x_i &= \bar{a}_{i0} - \sum_{j\in J} \bar{a}_{ij} s_j && i \in I \\ x_i &= 0 - (-1) s_i && i \in J \\ x_i &\geq 0, && i \in I \cup J, \end{aligned} \tag{3}$$

where I and J index the basic and nonbasic variables, respectively, and where the apex v^1 of C_1 is represented as the $(n+m)$-vector with components $\bar{a}_{i0}$, $i \in I$, and 0, $i \in J$. We will write this vector as $\bar{a}_0$, and the columns of the system (3) as $\bar{a}_j$, with components $\bar{a}_{ij}$ for $i \in I$, -1 for $i = j$, and 0 for $i \in J \setminus \{j\}$. Then the n extreme rays of C_1 are $r_j := \{x \in \mathbb{R}^{m+n} : x = \bar{a}_0 - \bar{a}_j s_j,\ s_j \geq 0\}$, $j \in J$.

Suppose that the halfspace H^+ with which we intersect C_1 to create the polyhedron C is $H^+ := \{x : x_h \geq 0\}$, where x_h is a basic variable. "Activating" H^+, i.e. replacing C_1 with $C := C_1 \cap H^+$ consists in pivoting x_h out of the basis in exchange for a nonbasic variable corresponding to one of the k extreme rays of C_1 that intersect H. An extreme ray r_j intersects H if and only if $\bar{a}_{hj} > 0$, and it intersects H before $\mathrm{bd}\, S$ if and only if the value of s_j for which x_h becomes 0, namely $\tilde{s}_j = \bar{a}_{h0}/\bar{a}_{hj}$, is less than the value of s_j for which r_j intersects $\mathrm{bd}\, S$. The intersection point $r_j \cap H$ is a new vertex of C. Notice that, if $\bar{a}_{h0} = 0$, i.e. the solution corresponding to v^1 is degenerate and $v^1 \in H$, then the "new" vertex of C coincides with v^1, i.e. no new vertex is in fact created.

Next we describe the procedure for generating the rays of C. For the sake of this discussion we assume that $S := \{x : 0 \leq x_q \leq 1\}$. The ray r_j intersects $H := \{x : x_h = 0\}$ only if $\bar{a}_{hj} > 0$. Also, r_j may intersect $\mathrm{bd}\, S$ either on the hyperplane $x_q = 0$, or on $x_q = 1$, depending on the sign of $\bar{a}_{qj}$. Thus r_j intersects H before $\mathrm{bd}\, S$ if and only if either

(i) $\bar{a}_{qj} > 0$ and $(\bar{a}_{h0}/\bar{a}_{hj}) < (\bar{a}_{q0}/\bar{a}_{qj})$

or

(ii) $\bar{a}_{qj} < 0$ and $(\bar{a}_{h0}/\bar{a}_{hj}) < (1 - \bar{a}_{q0})/(-\bar{a}_{qj})$.

In the case $\bar{a}_{qj} = 0$, r_j does not intersect $\mathrm{bd}\, S$ at all.

Consequently, we define the index of those rays r_j that intersect H before $\mathrm{bd}\, S$ as $J^+ := \{j \in J : \bar{a}_{hj} > 0 \text{ and either (i) or (ii) holds}\}$.

First, for every $j \in J^- := J \setminus J^+$, the ray $r_j := \{x : x = \bar{a}_0 - \bar{a}_j s_j,\ s_j \geq 0\}$ of C_1 is also a ray of C, hence its intersection point p^j with $\mathrm{bd}\, S$ is the same as q^j of

C_1. These intersection points p^j, $j \in J^-$, are readily available and do not require any pivots. Their set is denoted Q_1 $(=J^-)$.

Next, for every $j \in J^+$, let $r_j \cap H = \bar{a}_0 - \bar{a}_j \tilde{s}_j$, where $\tilde{s}_j = \bar{a}_{h0}/\bar{a}_{hj}$, and let the associated simplex tableau, obtained by pivoting on $\bar{a}_{hj}$, be

$$x_i = \tilde{a}_{i0} - \sum_{\ell \in J \setminus \{j\}} \tilde{a}_{i\ell} s_\ell - \tilde{a}_{ih} x_h \qquad i \in (I \setminus \{h\}) \cup \{j\}. \tag{4$_j$}$$

As soon as we have created the new vertex $\tilde{a}_0 = (\tilde{a}_{i0})$ (which in the case $v^1 \in H$ coincides with v^1), we delete the intersection point $q^j = r_j \cap \operatorname{bd} S$. If we adopt the notation of $C(v)$ for the cone with apex at v, then C_1 becomes $C(v^1)$ and the cone with apex at the new vertex $\tilde{a}_0$ is $C(\tilde{a}_0)$. Of the n extreme rays $r_\ell := \{x \in \mathbb{R}^{m+n} : x = \tilde{a}_0 - \tilde{a}_\ell s_\ell,\ s_\ell \geq 0\}$, $\ell \in (J \setminus \{j\}) \cup \{h\}$, of $C(\tilde{a}_0)$, k contain edges of C joining $\tilde{a}_0$ to $\bar{a}_0$ and to the $k-1$ other new vertices created by intersecting C_1 with H. These k extreme rays of $C(\tilde{a}_0)$ that contain (finite) edges of C are those r_ℓ such that $\ell \in (J^+ \setminus \{j\}) \cup \{h\}$. The remaining $n-k$ extreme rays r_ℓ of $C(\tilde{a}_0)$, intersected with $\operatorname{bd} S$, provide $n-k$ points p^j, $j \in Q_2$.

Note that after the pivot that takes us from $\bar{a}_0$ to $\tilde{a}_0$, one additional pivot is required for intersecting each extreme ray originating at $\tilde{a}_0$ with $\operatorname{bd} S$, hence a total of $1+(n-k)$ pivots for obtaining these $n-k$ new intersection points p^j.

Returning to the simplex tableau (3) and repeating this procedure for the remaining $k-1$ indices $j \in J^+$, we obtain for every such j a simplex tableau of the form $(4)_j$, and from it a set of $n-k$ new extreme rays r_ℓ, $\ell \in J^-$, and their intersection points with $\operatorname{bd} S$, $p^{j\ell}$, $\ell \in J \setminus J^+$. This gives a total of $k(n-k)$ new extreme rays of C, and their intersection points $p^{j\ell}$, $j \in J^+$, $\ell \in J^-$ with $\operatorname{bd} S$, at the total cost of $(k+1)(n-k)$ pivots. Together with the $n-k$ extreme rays that C inherits from C_1, and their intersection points with $\operatorname{bd} S$, this yields a total of $(k+1)(n-k)$ intersection points.

As to the actual expression for the intersection points, that is given by the value s_ℓ^* of s_ℓ for which r_ℓ intersects $\operatorname{bd} S$, which of course depends on the set S. If S is the intersection of two halfspaces like in the case of a split cut derived from $0 \leq x_q \leq 1$, the intersection with $\mathrm{bd}\, S$ of an extreme ray $r_\ell := \{x \in \mathbb{R}^{m+n} : x = \tilde{a}_0 - \tilde{a}_\ell s_\ell\}$ from a tableau $(4)_j$ is given by

$$s^*_\ell = \min\left\{\frac{\tilde{a}_{q0}}{\tilde{a}_{q\ell}}, \frac{1-\tilde{a}_{q0}}{-\tilde{a}_{q\ell}}\right\}.$$

The intersection of $\mathrm{bd}\, S$ with an extreme ray from the tableau (3) is given by an analogous expression.

4 Iterating the facet-activation

The procedure described above for "activating" a facet-defining halfspace H^+ of P by intersecting it with C_1, and intersecting the new extreme rays of the polyhedron C created this way with the boundary of S can of course be iterated. However, a condition for the iterated procedure to be valid is that the set S be kept unchanged.

The only difference between the first iteration, described in the previous section, and subsequent iterations, is that in subsequent iterations the hyperplane H being activated may intersect not only extreme rays, but also finite edges of C created during previous iterations. When this is the case, we can generate a new vertex as the intersection of H with the edge in question, treat this vertex the same way as in the first iteration, and remove all the intersection points cut off by H^+.

It is not hard to see that Theorems 1-6, stated for the set C obtained from C_1 by activating a single halfspace (inequality) H^+, also hold for the more general polyhedral set C obtained by iterating the procedure as described above. The proofs remain valid subject to the obvious minor changes implied by the iteration of the procedure for generating new rays.

At any point in the above procedure the set

$$\mathrm{Alpha}_Q := \{\alpha \in \mathbb{R}^n : \alpha p^j \geq \beta_1,\ j \in Q\}$$

is the reverse polar (upper polar) or the standard polar of the set $\{p^j\}_{j\in Q}$, depending on the sign of β_1. This means, among other things, that the vertices of Alpha_Q, i.e. the basic solutions of (1), correspond to facets of $\text{conv}\,\{p^j\}_{j\in Q}$. At any stage of the above procedure, the number of meaningful cuts obtainable from Alpha_Q is the number of basic solutions of $(1)_{\bar{\beta}}$, bounded by $\binom{|Q|}{n}$, the number of bases. In case we pursue the above procedure until we generate all intersection points of extreme rays of C with $\text{bd}\, S$, this number may of course become exponential in n.

5 A new cut generating paradigm

The approach described in the last two sections suggests a new cut generating paradigm, free of the curse of numerical difficulties associated with the dual degeneracy that accompanies iterative cut generation/reoptimization type procedures. Rather than getting to the deeper cuts by iterating the cut generation/reoptimization cycle, the new paradigm consists in generating intersection points of the edges of a relaxation C of P with the boundary of some cut generating set S, and using this collection of intersection points to generate deeper cuts without losing numerical accuracy.

In other words the approach envisaged here is to apply the procedure of sections 3-4 to generate a substantial number of intersection points of extreme rays of C with $\text{bd}\, S_k$ for several proper convex sets S_k, then formulate for each S_k a linear program $(\text{LP})_k$ with the constraint set $(1)_{\bar{\beta}}$ associated with S_k and possibly several objective functions. Each such linear program has n variables and a number of constraints that depends on how far we go in generating intersection points, and its solution α provides the coefficient vector of a generalized intersection cut $\alpha x \geq \bar{\beta}$ from S_k. The cuts obtained this way correspond to cuts that, if generated by a standard procedure, would come after many recursive applications of that procedure.

There are many ways in which the proposed new paradigm could be implemented, and choosing the best one requires experimentation. We are in the process of setting up these experiments and plan to report on their outcome in the near future.

6 Geometric interpretation of strengthened intersection cuts

It is well known that if $\alpha s \geq 1$ is the intersection cut (in the space of the nonbasic variables indexed by J) generated from the proper convex set S and the polyhedral cone C_1 with apex at $\bar{a}_0$ and extreme rays $r_j := \{x : x = \bar{a}_0 - \bar{a}_j s_j, s_j \geq 0\}$, $j \in J$, then $\alpha_j = \frac{1}{s_j^*}$, where

$$s_j^* = \max\{s_j : \bar{a}_0 - \bar{a}_j s_j \in S\}.$$

We will call $\alpha s \geq 1$ the *standard* intersection cut from C_1 and S. It is also well known that if some of the variables s_j, say those for $j \in J_1 \subseteq J$, are integer-constrained, then the intersection cut $\alpha s \geq 1$ can be strengthened by modularizing the $\bar{a}_j$, $j \in J_1$. Such techniques have been pioneered by Gomory and Johnson [11, 13] in a group theoretic framework, then pursued by Balas and Jeroslow [5] in a disjunctive programming context, and more recently revisited by Dey and Wolsey [9] and others in the context of intersection cuts from multiple inequalities.

We would like to find an analogous strengthening procedure for the generalized intersection cuts obtained via Theorem 4. But here we run into difficulties. The generalized intersection cut $\alpha x \geq \bar{\beta}$ of Theorem 4 is obtained as a basic feasible solution to the system $(1)_{\bar{\beta}}$, and it is not clear what the relationship of each coefficient α_j is to the columns of the system defining C. True, α is associated with a basis of $(1)_{\bar{\beta}}$ and hence with a set of n points p^j, $j = j_1, \ldots, j_n$. But on the one hand this connection is not trivial to specify; on the other, even if we were able to specify the connection and could consequently try to modularize the columns of the system defining C associated with each of the n points p^j, such an approach would still not work, for the obvious reason that the n modified points obtained from the p^j, say $\bar{p}^j$, $j = j_1, \ldots, j_n$ through the modularization, may not define a feasible basis, i.e. we have no reason to assume that the hyperplane through these n modified points $\bar{p}^j$ has all the remaining points

p^j (or their modularized correspondents $\bar{p}^j$) on one side.

It turns out, however, that these seemingly formidable difficulties can be avoided by using a different approach, based on a geometric interpretation of the strengthened intersection cuts obtained by modularization.

Suppose the variables s_j are integer-constrained for $j \in J_1 \subseteq J$, and we use this property to strengthen the standard intersection cut $\alpha s \geq 1$. Then the strengthened cut $\bar{\alpha} s \geq 1$ has coefficients

$$\bar{\alpha}_j = \begin{cases} \frac{1}{s_j^{**}} & j \in J_1 \\ \alpha_j & j \in J \setminus J_1, \end{cases}$$

where for $j \in J_1$,

$$s_j^{**} := \max\{s_j : \bar{a}_0 - \varphi(\bar{a}_j)s_j \in S\}$$

with $\varphi(\cdot)$ being the "strengthening" function whose exact definition depends on S. Thus, in the case of an intersection cut from $0 \leq x_k \leq 1$, $\varphi(\bar{a}_j)$ replaces $\bar{a}_{kj}$ with the optimal value of $\bar{a}_{kj}(\text{mod } 1)$ (which is $\bar{a}_{kj} - \lfloor \bar{a}_{kj} \rfloor$ if $\bar{a}_{kj} > 0$ and $\bar{a}_{kj} - \lceil \bar{a}_{kj} \rceil$ if $\bar{a}_{kj} < 0$) and leaves unchanged $\bar{a}_{ij}$ for $i \neq k$. In the case of an intersection cut from q rows of the simplex tableau indexed by Q, $\varphi(\bar{a}_j)$ replaces $\bar{a}_{ij}$ with $\bar{a}_{ij}(\text{mod } 1)$ for each $i \in Q$ and leaves unchanged $\bar{a}_{ij}$ for $i \notin Q$.

We will call $\bar{\alpha} s \geq 1$ the *strengthened* (as opposed to the standard) intersection cut from C_1 and S.

Now let $\bar{C}_1$ be the polyhedral cone with apex at $\bar{a}_0$ and extreme rays $\bar{r}_j := \{x : x = \bar{a}_0 - \varphi(\bar{a}_j)s_j, s_j \geq 0\}$, $j \in J_1$, $\bar{r}_j = r_j$, $j \in J \setminus J_1$.

$\bar{C}_1$, like C_1, is n-dimensional, since the vectors $\varphi(\bar{a}_j)$ are linearly independent. To see this, notice that the nonbasic components (i.e. the ones indexed by J) of each vector $\bar{a}_j$, $j \in J$, are

$$\bar{a}_{jk} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

and the same holds for $\varphi(\bar{a}_j)$, $j \in J$, i.e. the mapping φ preserves the independence of the $\bar{a}_j$, $j \in J$.

Theorem 7. *The strengthened intersection cut from C_1 and S is equivalent to the standard intersection cut from $\bar{C}_1$ and S.*

Proof. The standard intersection cut from $\bar{C}_1$ and S is $\beta s \geq 1$, where

$$\beta_j := \begin{cases} \frac{1}{s_j^{**}} & j \in J_1 \\ \frac{1}{s_j^*} & j \in J \setminus J_1. \end{cases}$$

Clearly, $\beta_j = \bar{\alpha}_j$ of (1), hence $\beta s \geq 1$ is the same as $\bar{\alpha} s \geq 1$, the strengthened intersection cut from C_1 and S. □

Consider now the set $\bar{C}$ constructed from $\bar{C}_1$ the way C is constructed from C_1. The only change in the procedure described in section 3 is that whenever a ray $r_j := \{x : \tilde{a}_0 - \tilde{a}_j s_j,\ s_j \geq 0\}$ is about to be intersected with $\operatorname{bd} S$, if $j \in N$, then r_j is replaced with its modularized counterpart $\bar{r}_j := \{x : x = \tilde{a}_0 - \varphi(\tilde{a}_j) s_j,\ s_j \geq 0\}$. Thus, as soon as the optimal LP solution $x = (\bar{a}_0, 0)$ is found, the cone $\bar{C}_1$ is substituted for C_1 by replacing each ray r_j, $j \in J_1$ with $\bar{r}_j$. Subsequently, whenever a halfspace is activated, for every newly created vertex v, those extreme rays r_j of the cone with apex at v that intersect $\operatorname{bd} S$ are replaced with their modularized counterparts $\bar{r}_j$.

Now let $\bar{r}_j$, $j \in \bar{Q}$, be the edges of $\bar{C}$ that intersect $\operatorname{bd} S$.

For $j \in \bar{Q}$, define $\bar{p}^j := \bar{r}_j \cap \operatorname{bd} S$, i.e.

$$\bar{p}^j := \begin{cases} \bar{a}_0 - \varphi(\bar{a}_j) s_j^{**} & j \in \bar{Q} \cap N_1 \\ \bar{a}_0 - \bar{a}_j s_j^* & j \in \bar{Q} \setminus N_1 \end{cases}$$

Furthermore, let $\bar{q}^j$, $j = 1, \ldots, n$, the intersection points of the extreme rays of $\bar{C}_1$ with $\operatorname{bd} S$, be defined analogously to $\bar{p}^j$. Then we have

Corollary 8. *Theorem 3 is valid with α^1 and p^j, $j \in Q$, replaced by $\bar{\alpha}^1$ and $\bar{p}^j$, $j \in \bar{Q}$.*

Corollary 9. *Theorem 4 is valid with α^1, p^j and q^j replaced by $\bar{\alpha}^1$, $\bar{p}^j$, $j \in \bar{Q}$ and $\bar{q}^j$, $j = 1, \ldots, n$, respectively.*

The upshot of Theorem 7 and its Corollaries is that instead of strengthening a given set of generalized intersection cuts, we ought to generate directly what we shall call strengthened (as opposed to standard) generalized intersection cuts. To accomplish this, we have to apply a slightly modified version of the procedure of section 3 for generating the rays of C and their intersection points with $\operatorname{bd} S$. The modification consists in the fact that instead of C we consider $\bar{C}$, the polyhedron obtained from C by replacing every infinite edge $r_j := \{x : x = \tilde{a}_0 - \tilde{a}_j s_j, s_j \geq 0\}$, such that $j \in N_1$, with an edge $\bar{r}_j := \{x : x = \tilde{a}_0 - \varphi(\tilde{a}_j) s_j, s_j \geq 0\}$. Otherwise the procedure remains unchanged.

7 The properties of generalized intersection cuts

As before, let $\mathcal{A}(P)$ be the hyperplane arrangement associated with P, and denote by $\operatorname{vert}(\mathcal{A}(P))$ the set of vertices of $\mathcal{A}(P)$. Let S be a proper convex set, let $V(S) := \operatorname{vert} \mathcal{A}(P) \cap \operatorname{int} S$, and let p^j, $j \in Q_S$, be the intersection points with $\operatorname{bd} S$ of the edges of P with only one end in $V(S)$. Finally, let $\mathcal{P}_S := \{p^j\}_{j \in Q_S}$.

We restate the definition of closure of P under a specified cut generating procedure or family of cuts. We define the *S-closure of P* as the set of points in P that satisfy all intersection cuts from S and the cones with apex in $V(S)$.

Theorem 10. *The solution set of all inequalities $\alpha x \geq \beta$ such that $\alpha p^j \geq \beta$, $j \in Q_S$ for $\beta \in \{1, 0, -1\}$ and $\alpha v < \bar{\beta}$ for some $v \in V(S)$ is the S-closure of P.*

Proof. Let $\tilde{\alpha} x \geq \tilde{\beta}$ be any intersection cut generated from some vertex in $V(S)$ and the convex set S. From Theorem 4, $\tilde{\alpha} x \geq \tilde{\beta}$ is dominated by $\bar{\alpha} x \geq \bar{\beta}$ for any basic optimal solution $\bar{\alpha}$ to the system $(1)_{\bar{\beta}}$, (2), such that $\bar{\alpha} v < \bar{\beta}$ for some $v \in V(S)$, where $\bar{\beta} = \tilde{\beta}$. Hence the solution set of all inequalities $\alpha x \geq \beta$ such that $\alpha p^j \geq \beta$, $j \in Q_S$ for $\beta \in \{1, 0, -1\}$, is contained in the solution set of all intersection cuts from S. □

If σ is a collection of convex sets S_i whose interior contains no integer point of P, then the *σ-closure of P* is the intersection of all S_i-closures, $S_i \in \sigma$. If σ is the set of all split disjunctions, then the σ-closure is the split closure [8].

Corollary 11. *Let $\mathcal{C}(S_i)$ be the set of inequalities whose solution set is the S_i-closure of P. Then the solution set of $\bigcup_{S_i \in \sigma} \mathcal{C}(S_i)$ is the σ-closure of P.*

If σ is the collection of all $S_i := \{x : \pi_0 \leq \pi x \leq \pi_0 + 1,\ (\pi, \pi_0) \in \mathbb{Z}^{n+1}\}$ containing some $\bar{x} \in V(S_i)$, then this set is the split closure of P.

In light of the results of section 5, Theorem 10 and Corollary 11 also apply to strengthened intersection cuts. Thus, we have

Theorem 10a. *The solution set of all inequalities $\bar{\alpha}x \geq \beta$ such that $\bar{\alpha}\bar{p}^j \geq \beta$, $j \in \bar{Q}_S$ for all $\beta \in \{1, 0, -1\}$, and $\bar{\alpha}v < \bar{\beta}$ for some $v \in V(S)$, is the S-closure of P with respect to strengthened intersection cuts.*

Thus we have a new expression for the split closure, and a new way of generating it. Next we proceed to compare it to the earlier known expression. This can best be done by considering the S_k-*closure of* P, i.e. the set defined by all intersection cuts from $S_k := \{x \in \mathbb{R}^n : 0 \leq x_k \leq 1\}$. Consider the following three families of cuts valid for $P_k := \{x \in P : x_k \leq 0 \ \vee \ x_k \geq 1\}$:

$F_1 := \{\ \alpha x \geq \beta : \alpha x \geq \beta$ is the intersection cut from S_k, where $x_k = a_{k0} - \sum_{j \in J} a_{kj} x_j$ and J is the nonbasic index set associated with a basic (feasible or infeasible) solution to P such that $0 < a_{k0} < 1\}$.

$F_2 := \{\ \alpha x \geq \beta : (\alpha, \beta, u, v, u_0, v_0)$ is a basic feasible solution to the cut generating linear program $(\text{CGLP})_k$ of the lift-and-project method [3]$\}$.

$F_3 := \{\ \bar{\alpha}x \geq \bar{\beta} : \bar{\alpha}$ is a basic feasible solution to $(1)_{\bar{\beta}}$ defined with respect to S_k for some $\bar{\beta} \in \{0, 1, -1\}$, satisfying $\bar{\alpha}v < \bar{\beta}$ for some $v \in V(S_k)\}$

Then the S_k-closure of P is the set of those $x \in \mathbb{R}^n$ satisfying all cuts in F_i where i can be either 1, 2 or 3.

The relationship between the cuts in F_1 and F_2 is well researched: there is a one-to-one correspondence between cuts in F_1 and F_2 [6]. However, the relationship between F_1 and F_2 on the one hand, and F_3 on the other, is different: in general, F_3 is a proper subset of F_1 (of F_2).

To see this, assume that P is not a simplicial cone, that $P \not\subseteq \{x : x_k \leq 0\}$,

$P \not\subseteq \{x : x_k \geq 1\}$, and that $(1)_{\bar{\beta}}$ has at least one solution that satisfies (2). Then we have

Theorem 12. *Every inequality of F_3 defines a facet of the S_k-closure of P, hence in general $F_3 \subsetneq F_1(= F_2)$.*

Proof. Every inequality of F_3, corresponding to a basic feasible solution $\bar{\alpha}$ of $(1)_{\bar{\beta}}$, is satisfied at equality by n affinely independent points p^j. As each of these points belongs to S_k-closure of P, the inequality defines a facet of the S_k-closure. Thus, except in the special case where F_1 $(= F_2)$ contains only facet defining inequalities, F_3 is a strict subset of $F_1(= F_2)$. □

Next we identify a class of intersection cuts, i.e. members of F_1 (of F_2), that are not members of F_3. This class consists of all intersection cuts from a cone C_1 with apex at a vertex v of P and a convex set $S := \{x \in \mathbb{R}^n : \pi_0 \leq \pi x \leq \pi_0 + 1\}$, where $(\pi_1, \pi_0) \in \mathbb{Z}^{n+1}$, such that $\pi_0 < \pi v < \pi_0 + 1$, that satisfy a certain requirement.

Let $\alpha^1 x \geq \beta_1$ be any intersection cut from S and a cone C_1 with apex at $v \in (\operatorname{vert} P) \cap \operatorname{int} S$, and let $\mathcal{T} := \{q^j\}_{j=1,\ldots,n}$ be the n intersection points of the extreme rays of C_1 with $\operatorname{bd} S$. Further, let $\operatorname{bd} S = S_0 \cup S_1$, where $S_0 := \{x : \pi x = \pi_0\}$, $S_1 := \{x : \pi x = \pi_0 + 1\}$; let $\mathcal{T}_0 := \mathcal{T} \cap S_0$, $\mathcal{T}_1 : \mathcal{T} \cap S_1$, and assume $\mathcal{T}_0 \neq \emptyset \neq \mathcal{T}_1$. Let C be the convex polyhedron defined by a subset of the inequalities defining P (possibly $C = P$), such that the vertex v of P (the apex of C_1) is also a vertex of C. Finally, let $\mathcal{P} := \{p^j\}_{j \in Q}$ be the set of intersection points of the edges of C with $\operatorname{bd} S$.

From Theorem 3, α^1 is a feasible solution to the system $(1)_{\bar{\beta}}$. Thus α^1 is not a member of F_3 if and only if it is not basic. The next theorem gives a necessary and sufficient condition for α^1 to be basic, which shows how seldom this condition is satisfied.

Theorem 13. *α^1 is a basic feasible solution of $(1)_{\beta_1}$ if and only if*

$$\dim(\operatorname{conv} \mathcal{P} \cap \operatorname{conv} \mathcal{T}_k) = \dim(\operatorname{conv} \mathcal{T}_k), \quad k = 0, 1$$

Proof. Sufficiency. Suppose the condition holds. Then for $k = 0, 1$, $\operatorname{conv} \mathcal{P} \cap \operatorname{conv} \mathcal{T}_k$ contains the same number of affinely independent points as $\operatorname{conv} \mathcal{T}_k$, say t_k; and since $t_0 + t_1 = n$, it follows that $\operatorname{conv} \mathcal{P} \cap (\operatorname{conv} \mathcal{T}_0 \cup \operatorname{conv} \mathcal{T}_1)$ contains n affinely independent points $p^j \in \mathcal{P}$ that satisfy $\alpha^1 p^j = \beta_1$. To see why this is so, notice that the points p^j in question are of two kinds: intersection points $p^j = q^\ell$ of some extreme ray q^ℓ of C_1 with $\operatorname{bd} S$, or intersection points $p^j \neq q^\ell$, $\forall \ell$, of some edge of C, other than those of C_1, with $\operatorname{bd} S$. Now the points $p^j = q^\ell$ satisfy $\alpha^1 p^j = \beta_1$ by the definition of the points q^ℓ, and the points $p^j \neq q^\ell$, $\forall \ell$, are convex combinations of the extreme points q^ℓ of the set $\operatorname{conv} \mathcal{T}_0$ or $\operatorname{conv} \mathcal{T}_1$ to which they belong, hence they also satisfy $\alpha^1 p^j = \beta_1$. Since α^1 satisfies $(1)_{\beta_1}$ and satisfies at equality a full rank subsystem of $(1)_{\beta_1}$, it is a basic feasible solution of $(1)_{\beta_1}$.

Necessity. Suppose the condition of the theorem is violated for $k = 0$ (the same argument holds for $k = 1$). Then the dimension of $\operatorname{conv} \mathcal{P} \cap (\operatorname{conv} \mathcal{T}_0 \cup \operatorname{conv} \mathcal{T}_1)$ is strictly less than n. This is so because no matter how many points of $\mathcal{P}$ are contained in $\operatorname{conv} \mathcal{T}_0$, the maximum number of affinely independent points in the set $\operatorname{conv} \mathcal{P} \cap \operatorname{conv} \mathcal{T}_0$ is given by its dimension. Thus the maximum number of affinely independent points in $\operatorname{conv} \mathcal{P} \cap (\operatorname{conv} \mathcal{T}_0 \cup \operatorname{conv} \mathcal{T}_1)$ is strictly less than n and hence α^1 is a nonbasic feasible solution of $(1)_{\beta_1}$. □

*

In this paper we have introduced a new framework for generating and using cuts. Instead of generating cuts and adding them to the linear relaxation, then pivoting to obtain deeper cuts, in this framework we generate intersection points of extreme rays of the linear relaxation with the boundary of a cut generating convex set. We then use these intersection points to generate the deepest cuts they can provide, in various directions, as needed. These generalized intersection cuts dominate many of the standard intersection cuts. There are in principle many ways in which this framework can be used, and determining the best ways requires further study and experimentation.

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