A Catalog of Minimally Nonideal Matrices

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August, 1995 (revised February, 1997)

Abstract

This paper describes a backtracking algorithm for the enumeration of nonisomorphic minimally nonideal $(n \times n)$ matrices that are not degenerate projective planes. The application of this algorithm for $n \leq 12$ yielded 20 such matrices, adding 5 matrices to the 15 previously known. For greater dimensions, $n = 14$ and $n = 17$, 13 new matrices are given. For nonsquare matrices, 38 new minimally nonideal matrices are described.

1 Introduction

A $(0,1)$-matrix A is *ideal* if the vertices of the polyhedron $Q(A) := \{x \in R^n | A \cdot x \geq 1, x \geq 0\}$ are integral. These matrices are also called *width-length matrices* [8][9], matrices with the *weak max-flow min-cut property* [17] or matrices with the *max-flow min-cut property* [2][8][9][18], but we follow the terminology of Cornuéjols and Novick [1] to stress the analogy with the well-known class of perfect matrices. (The reader is referred to [12] for the definitions and concepts related to polyhedral theory.)

A $(0,1)$-matrix A is *minimally nonideal (mni)* if and only if the three following conditions are met:

(i) A does not contain a dominating row;

(ii) $Q(A)$ is not an integral polyhedron;

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*Research supported by FNRS grant, Switzerland, while visiting Carnegie Mellon University, Pittsburgh, PA 15213.

(iii) for all $i \in \{1, \ldots, n\}$, the two polyhedra

$$Q_{x_i=0}(A) := \{x \in R^n | A \cdot x \geq 1, x \geq 0, x_i = 0\} \quad \text{and}$$

$$Q_{x_i=1}(A) := \{x \in R^n | A \cdot x \geq 1, x \geq 0, x_i = 1\}$$

are integral.

Recall that a $(0,1)$-vector v in R^n *dominates* a $(0,1)$-vector w in R^n if $v_k \geq w_k$ for all $1 \leq k \leq n$. A $(0,1)$-matrix A contains a *dominating row* if there exist two distinct rows A_i and A_j of A such that A_i dominates A_j.

The starting point of the study of mni matrices is Lehman's work [8][9]. It contains a wonderful characterization of these matrices for which Seymour [18] and Padberg [14] gave alternative proofs. Lehman [8] cites three infinite classes of mni matrices, the circulant matrices C_n^2 and their blockers $b(C_n^2)$ for all odd $n \geq 3$ and the degenerate projective planes J_s for all $s \geq 2$. He also mentions another mni matrix that does not belong to any of these classes, the matrix F_7 corresponding to the points and lines of the Fano plane. (The definition of blocker, circulant matrix and degenerate projective plane is given in the next section).

Several other mni matrices have been found since: Seymour [17] noted that the matrix $O(K_5)$ whose columns correspond to the edges of the complete graph on 5 nodes K_5 and whose rows are the incidence vectors of the odd cycles of K_5 is an mni matrix. Cornuéjols and Novick [1] observed that the row submatrix $\tau(K_5)$ of $O(K_5)$ whose rows are the incidence vectors of the triangles of K_5 is also an mni matrix and were able to construct 1474 nonisomorphic mni matrices by adding rows to $\tau(K_5)$. Ding [2] found an (8×8) mni matrix called D_8. An important contribution came from Cornuéjols and Novick [1], as they were able to give the list of all mni circulant matrices, adding 8 mni matrices in the process (two of which were found independently by Qi [15] and one by Ding [2]).

Nevertheless, we lack a good understanding of the structure of mni matrices and, as a consequence, of ideal matrices. The aim of this paper is to make easier the study of mni matrices by providing a significant number of them. We first describe a backtracking algorithm for the enumeration of the potential cores of mni matrices (see the next section for the definition of the core) and report mni matrices obtained from these cores, namely:

- all square mni matrices up to dimension (12×12), adding 5 new mni matrices to the 15 previously known ones;
- 13 new square mni matrices of dimensions (14×14) and (17×17);
- 38 new nonsquare mni matrices with 11, 14 and 17 columns with nonisomorphic cores.

The enumeration of the potential cores is performed by first enumerating, for fixed values of n and k, all nonisomorphic $(n \times n)$ $(0,1)$-matrices with exactly k 1's per row and per column. Such a matrix may be interpreted as the adjacency matrix of a bipartite graph where each vertex has degree k (see Section 4 for details). In other words, the algorithm generates all *nonisomorphic* k-regular bipartite graphs on a fixed number of vertices. We stress the term nonisomorphic since very few graph enumeration problems are amenable to a successful generation when nonisomorphic graphs are sought.

2 Lehman matrices

Relatively few results related to mni matrices are known, the most important ones are due to Lehman [8][9], Padberg [14], Seymour [18], Novick [13] and Cornuéjols and Novick [1]. In particular, Theorem 2.5 given below states a beautiful property of these matrices. In this section we first review some of these results. Then we define a class of matrices, the Lehman matrices, related to the mni matrices and give some of their properties.

A matrix A is *isomorphic* to a matrix B if B may be obtained by a permutation of the rows followed by a permutation of the columns of A.

For all integer $s \geq 2$, the *degenerate projective plane* J_s is the square matrix of dimension $(s+1) \times (s+1)$ whose columns are indexed by $\{0, \dots, s\}$ and whose rows are the incidence vectors of the sets $\{1, \dots, s\}$, $\{0,1\}, \{0,2\}, \dots, \{0,s\}$.

It is easy to check that for all $s \geq 2$, J_s is an mni matrix and thus the degenerate projective planes form an infinite class of mni matrices. These matrices are also particular among the mni matrices, as all mni matrices that are not isomorphic to degenerate projective planes share several properties.

Theorem 2.1 *[9][14] If A is an mni matrix then A is isomorphic to either*

(i) the degenerate projective plane J_s, $s \geq 2$, or

(ii) $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where A_1 is a square nonsingular matrix with $r \geq 2$ '1's per row and per column, and each row of A_2 has at least $(r+1)$ '1's.

Let A be a $(0,1)$-matrix with n columns. If A contains an $(n \times n)$ row submatrix A_1 such that A_1 is nonsingular, the number of '1's in each row and in each column of A_1 is $r \geq 2$ and the number of '1's in each row of A that is not in A_1 is strictly greater than r, then A_1 is called the *core* of A.

Note that A contains at most one core and if A is an mni matrix that is not isomorphic to a degenerate projective plane, then Theorem 2.1 implies that A contains a core.

A $(0,1)$-vector v is *minimal* for a property $\mathcal{P}$ if v has the property $\mathcal{P}$ but each vector v' obtained by switching one or several entries '1' of v to '0' does not have the property. Let A

be a $(0,1)$-matrix with n columns. The *blocker* $b(A)$ of A is the matrix with n columns and whose rows are the minimal $(0,1)$-vectors v such that $A \cdot v \geq 1$.

Lemma 2.2 *[8] A $(0,1)$-matrix A is ideal if and only if $b(A)$ is ideal.*

Corollary 2.3 *A $(0,1)$-matrix A without dominating row is an mni matrix if and only if $b(A)$ is an mni matrix.*

It follows from the corollary that Theorem 2.1 holds for $b(A)$ as well, i.e. $b(A) = \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right]$ where B_1 is the core of $b(A)$ with $s \geq 2$ '1's per row and per column, and each row of B_2 has at least $(s+1)$ '1's. We denote by E_n the $(n \times n)$ matrix with all entries equal to '1' and by I_n the $(n \times n)$ identity matrix.

For $1 \leq k \leq n$, let C_n^k denote the $(n \times n)$ matrix with entries

$$c_{ij} = \begin{cases} 1 & \text{for } \ j-1 = (i-1),\ i, \ldots,\ (i+k-2) \ \ \text{mod n} \\ & \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \text{for} \qquad i = 1, \ldots, n.$$

The matrices C_n^k are called *circulant matrices.* For odd $n \geq 3$ the circulant matrices C_n^2 are mni matrices. Thus C_n^2 and their blockers $b(C_n^2)$, for odd $n \geq 3$, form two other infinite classes of mni matrices. The core of a matrix $b(C_n^2)$, for odd $n \geq 3$, is $C_n^{(n+1)/2}$. Cornuéjols and Novick have conjectured that C_n^2 and $C_n^{(n+1)/2}$ are essentially the only cores contained in the mni matrices:

Conjecture 2.4 *[1] There exists an n_0 such that, except for the degenerate projective planes, each mni matrix with $n \geq n_0$ columns has a core isomorphic to C_n^2 or $C_n^{(n+1)/2}$.*

They also showed that the conjecture is true (with $n_0 = 18$) for circulant matrices [1].

Theorem 2.5 *[9][14][18] A matrix A with n columns is an mni matrix if $Q(A)$ has a unique fractional extreme point and either*

(i) A is isomorphic to the degenerate projective plane J_{n-1}, or

(ii) $A_1 \cdot B_1^T = E_n + (r \cdot s - n) I_n$

where A_1 is the core of A, B_1 is a matrix obtained by a suitable permutation of the rows of the core of $b(A)$ and $r \geq 2$ (resp. $s \geq 2$) is the number of '1's per row and per column of A_1 (resp. B_1).

A matrix without dominating row, such that both A and $b(A)$ contain a core and satisfy condition (ii) of Theorem 2.5 is a *Lehman matrix.*

Theorem 2.5 shows that mni matrices not isomorphic to a degenerate projective plane are Lehman matrices. This result is of interest, as no polynomial time method is known to check if a matrix A is an mni matrix. We will see below that recognizing a Lehman matrix may be done in polynomial time. Thus it may be computationally less expensive to test if A is a Lehman matrix and only if this is the case to test if A is mni. The efficiency of this scheme depends of course on the number of Lehman and mni matrices. Before addressing this question, we give elementary results on the structure of $Q(A)$ and $Q(A_1)$ when A is a Lehman matrix and show that the converse of Theorem 2.5 holds. A first useful observation is that the integers r, s and n in Theorem 2.5 are not arbitrary:

Lemma 2.6 *[14] Let A be a Lehman matrix, let A_1 be its core with r '1's per row and per column and let B_1 be the core of $b(A)$ with s '1's per row and per column. Then $1 \leq r \cdot s - n \leq min\ (r-1, s-1)$.*

A direct consequence of Lemma 2.6 is that there exists no Lehman matrix with 6 or 12 columns and thus no mni matrix with 6 or 12 columns.

Lemma 2.7 *Let A be a Lehman matrix with n columns such that its core A_1 has r '1's per row and per column. Then*

(i) $f = (\frac{1}{r}, ..., \frac{1}{r})$ is a fractional extreme point of $Q(A)$ (resp. $Q(A_1)$);

(ii) f is incident to exactly n edges of $Q(A)$ (resp. $Q(A_1)$) joining it to the n integral extreme points $b_1, \ldots, b_n$;

(iii) the n extreme points $b_1, \ldots, b_n$ of $Q(A)$ (resp. $Q(A_1)$) are rows of the blocker $b(A)$ (resp. $b(A_1)$);

(iv) A_1 is a Lehman matrix.

Proof. As the matrix A_1 has exactly $r \geq 2$ '1's per row, we have $A_1 \cdot f = 1$ and thus f is in $Q(A_1)$. As each row of A that is not in A_1 contains $> r$ '1's, we have $A \cdot f \geq 1$ showing that f is also in $Q(A)$. Moreover, as A_1 is nonsingular by the definition of the core, f is an extreme point of both $Q(A)$ and $Q(A_1)$, yielding (i).

Let A_1^i be the matrix obtained by deleting row i of A_1. Condition (ii) of Theorem 2.5 can be stated as: The rows $b_1, \ldots, b_n$ of the core B_1 of the blocker $b(A)$ may be numbered such that, for all $i = 1, \ldots, n$, $a_i \cdot b_i > 1$ and $A_1^i \cdot b_i = 1$. In other words, each of the n points b_i lies on an edge of $Q(A)$ incident to f. Since the rows of A_1 are linearly independent and the other rows in A have more than r '1's, exactly n edges of $Q(A)$ are incident to f. Moreover, each b_i is an extreme point of $Q(A)$ since A_1^i has rank $(n-1)$, b_i has at least one entry, say entry

$k(i)$, equal to 0 and thus b_i is the unique solution to the system $A_1^i \cdot x = 1$, $x_{k(i)} = 0$, proving (ii) and (iii) for $Q(A)$. Similar arguments show that exactly n edges of $Q(A_1)$ are incident to f and that these n edges join f to the n extreme points $b_1, \ldots, b_n$ of $Q(A_1)$.

As B_1 is the core of $b(A)$ we have that the n points $b_1, \ldots, b_n$ satisfy $1 \cdot x = s$, where $s \geq 2$ is the number of '1's per row in B_1. Hence, the extreme points of the polytope given by $S := \{x \in R^n | A_1 \cdot x \geq 1, x \geq 0, 1 \cdot x \leq s\}$ are $f, b_1, \ldots, b_n$. It follows that the only integral points contained in S are $b_1, \ldots, b_n$. Thus, each row of $b(A_1)$ that is not in B_1 contains at least $(s + 1)$ '1's, i.e. B_1 is the core of $b(A_1)$, yielding (iii) for A_1 and (iv). ∎

Points (iii) and (iv) of the previous lemma may be a little bit disappointing, as they imply that if A_1 is a square Lehman matrix with r '1's per row and B_1 is the core of $b(A_1)$, then adding to A_1 any collection of $(0, 1)$-vectors a with at least $(r+1)$ '1's and such that $a \cdot B_1^T \geq 1$ yields a Lehman matrix A as soon as A does not contain a dominating row. It seems very unlikely that all these matrices are mni matrices which leads to the conclusion that the number of Lehman matrices is much higher than the number of mni matrices. On the other hand, the fact that the core of any mni matrix is a Lehman matrix is of interest: The enumeration of all mni matrices may be split into two distinct subproblems. The first one is to enumerate all square Lehman matrices. The second one is to determine all collections of rows that can be added to a given square Lehman matrix such that the resulting matrix is mni.

In this paper, we will essentially address the first of these two problems. We will deal partially with the second one in looking for only one collection of rows that can be added to a given square Lehman matrix to obtain an mni matrix. The motivation for the latter problem arises from Conjecture 2.4 and Theorem 2.5. The theorem induces a classification of mni matrices by grouping matrices with isomorphic cores. It is then natural to try to determine all possible nonisomorphic cores that may be completed to mni matrices.

Theorem 2.5 states that an mni matrix A that is not isomorphic to a degenerate projective plane is a Lehman matrix such that $Q(A)$ has a unique fractional extreme point. In fact, the converse also holds, an observation justifying the validity of the completion algorithm presented in Section 6:

Lemma 2.8 *Let A be a Lehman matrix such that $Q(A)$ has a unique fractional extreme point. Then A is an mni matrix.*

Proof. Let $i \in \{1, \ldots, n\}$. We have to show that the polyhedra $Q_{x_i=0}(A)$ and $Q_{x_i=1}(A)$ are both integral. W.l.o.g, we assume that $i = 1$. Note that Lemma 2.7 shows that the unique fractional extreme point of $Q(A)$ is $f = (\frac{1}{r}, \ldots, \frac{1}{r})$. Thus $Q_{x_1=0}(A)$ is a face of $Q(A)$ that does not contain f, implying the integrality of this polyhedron.

Polyhedron $Q_{x_1=1}(A)$ is obtained by adding the equality $x_1 = 1$ to $Q(A)$. Hence, an extreme point q of $Q_{x_1=1}(A)$ is either an extreme point p of $Q(A)$ with $p_1 = 1$ (all these points are integral by hypothesis) or q is obtained from the intersection of an edge of $Q(A)$ with the hyperplane $x_1 = 1$. As all extreme points of $Q(A)$ satisfy $x_1 \leq 1$, such an edge is incident to an extreme point y of $Q(A)$ with $y_1 < 1$ and its direction is an extreme ray of $Q(A)$. By Lemma 2.7, the only fractional extreme point of $Q(A)$ is f and every edge of $Q(A)$ incident to f is incident to another extreme point of $Q(A)$. Hence $f \neq y$, implying that y is integral and $y_1 = 0$. As the extreme rays of $Q(A)$ are the n unit vectors e^i for $i = 1, \dots, n$, the direction of the edge is e^1 and the intersection of this edge with the hyperplane $x_1 = 1$ is the integral point $q = (1, y_2, y_3, \dots, y_n)$. ∎

3 Recognizing Lehman matrices

The proof of Lemma 2.7 and the following theorem yield a polynomial algorithm to test if a given matrix A with n columns is a Lehman matrix. A first observation is that testing if A contains a core is easy: Let $r \geq 2$ be the minimum number of '1's in a row of A. Let A_1 be the submatrix of A containing all rows of A with exactly r '1's. If A_1 is not a square matrix or if A_1 has not exactly r '1's per column or if A_1 is a square and singular matrix, then A has no core and thus is not a Lehman matrix. Moreover, checking that A does not contain a dominating row is also trivial. It remains to show how to decide if a matrix A, containing a core A_1 and no dominating row, is a Lehman matrix.

Theorem 3.1 *Let A be a $(0,1)$-matrix with n columns, no dominating row and containing a core A_1 with $r \geq 2$ '1's per row and per column. Then A is a Lehman matrix if and only if the n extreme points $b_1, \dots, b_n$ adjacent to the extreme point $f = (\frac{1}{r}, \dots, \frac{1}{r})$ in $Q(A)$ have only $\{0,1\}$ entries and there exists a matrix B_1 having $b_1, \dots, b_n$ as rows such that B_1 has exactly s '1's per row and per column and $A_1 \cdot B_1^T = E_n + (r \cdot s - n)I_n$.*

Proof. Note that, since A contains a core A_1, then f is an extreme point of $Q(A)$ and exactly n edges of $Q(A)$ are incident with f. These edges may be determined by solving the n linear systems $A_1^i \cdot x = 1$, where A_1^i is the matrix obtained by deleting row i of A_1, for $i = 1, \dots, n$.

If A is a Lehman matrix, the result follows directly from Lemma 2.7. Conversely, if there exists $b_1, \dots, b_n$ and a matrix B_1 with the given properties, then b_i is a solution to $A_1^i \cdot x = 1$, for $i = 1, \dots, n$. By definition, A is a Lehman matrix if all the points $b_1, \dots, b_n$ are rows of $b(A)$ and each of the remaining rows of $b(A)$ has at least $(s+1)$ '1's.

As the n points $b_1, \dots, b_n$ satisfy $1 \cdot x = s$, the extreme points of the polytope given by $S := \{x \in R^n | A \cdot x \geq 1, x \geq 0, 1 \cdot x \leq s\}$ are $f, b_1, \dots, b_n$. It follows that the integral points contained in S are $b_1, \dots, b_n$. Thus, each row of $b(A)$ that is not in B_1 contains at least $(s+1)$ '1's, i.e. B_1 is the core of $b(A)$. ∎

It follows that whether a matrix A is a Lehman matrix or not depends for a large part on the polyhedron $Q(A_1)$ associated with its core A_1: For all $1 \leq i \leq n$, first check if the linear system $A_1^i \cdot x = 1$ has a solution b_i contained in $Q(A_1)$ with only $\{0,1\}$ entries. This is easy, as the solution set S of the linear system is given by $S = \{s(\lambda) = f + \lambda \cdot v^i, \lambda \in R\}$ where $v^i \neq 0$ is a solution to $A_1^i \cdot x = 0$. Moreover, as f is an extreme point of $Q(A_1)$, $s(\lambda)$ is not in $Q(A_1)$ either if $\lambda > 0$ or if $\lambda < 0$ and thus there exists at most one value of λ such that $s(\lambda)$ has only $\{0, 1\}$ entries. If one of $b_1, \ldots, b_n$ does not exist or is not in $Q(A)$, then A is not a Lehman matrix. Otherwise, construct the matrix B_1 with b_i as ith row, for $i = 1, \ldots, n$ and check that this matrix has s '1's per row and per column and that $A_1 \cdot B_1^T = E_n + (r \cdot s - n)I_n$. This is the case if and only if A is a Lehman matrix.

4 Enumeration of square Lehman matrices

An $(0, 1)$-matrix A is an *(n, r)-regular matrix* if A is $(n \times n)$ and has r '1's per row and per column. In this section, we present an algorithm for enumerating nonisomorphic (n, r)-regular matrices. Then, we determine which of these matrices are Lehman matrices using the recognition algorithm outlined in Section 3.

An enumeration problem consists in enumerating all objects satisfying a given property $\mathcal{P}$. For most properties of interest, for example "to be a vertex of the polyhedron $\{x \in R^n | A \cdot x \geq b\}$" or "to be a spanning tree of the graph G = (V, E)", there are instances for which the number of objects to be listed is exponential in the input size. As any algorithm has a time complexity at least linear in the required output size, the usual definition of a polynomial time algorithm is not very meaningful for this kind of problems. Instead, a *linear time* algorithm for an enumeration problem is an algorithm whose time complexity is bounded by the output size times a polynomial in the input size. Using the language of Valiant [19], the existence of a linear-time algorithm is equivalent to the *P-enumerability* for property $\mathcal{P}$.

Another important concept is the space complexity of an enumeration algorithm. Here again, as the number of objects to be found may be very large, algorithms which do not require the storage of these objects are usually preferred. An algorithm for an enumeration problem is *compact* if its space complexity is bounded by a polynomial in the input size. An enumeration problem is *strongly P-enumerable* if there exists a compact linear time algorithm to solve it. We present below a compact algorithm for enumerating nonisomorphic (n, r)-regular matrices.

An *(n, n)-bipartite graph* is a bipartite graph on $2n$ nodes with n nodes on each side of the bipartition. A graph is r-regular if the degree of each node is exactly r. It is well-known that the $(n \times n)$ $(0, 1)$-matrices are in one-to-one correspondence with the (n, n)-bipartite graphs: For $i = 1, \ldots, n$, associate a node i (resp. i') to the ith row (resp. column) of the matrix and an edge (i, j') exists in the graph if and only if the entry (i, j') of the matrix is '1'. Thus, enumerating the (n, r)-regular matrices is equivalent to the enumeration of the r-regular (n, n)-bipartite graphs. For any $r \geq 1$, there exists a compact linear time algorithm for enumerating all r-regular (n, n)-bipartite graphs contained in a given (n, n)-bipartite graph G (see [5] for the case $r = 1$ and a similar algorithm may be obtained for larger r [4]).

Here, we are not interested in getting all (n, r)-regular matrices, but rather in obtaining a maximal set of nonisomorphic (n, r)-regular matrices. Introducing conditions of nonisomorphism for a graph enumeration problem usually implies an increase in difficulty although the output size is drastically reduced. This is the case for the enumeration of the r-regular nonisomorphic bipartite graphs, except for the cases $r = 1, 2$ that are trivial.

The direct approach consisting in enumerating all (n, r)-regular matrices and removing all but one in each isomorphism class is not very appealing due to the huge amount of wasted time handling isomorphic matrices. Instead, we describe below a simple backtracking approach that avoids as much as possible the generation of isomorphic matrices, yielding a compact algorithm for the enumeration of the nonisomorphic (n, r)-regular matrices.

Let A be an $(m \times n)$ $(0, 1)$-matrix. The binary number obtained by concatenation of the rows $A_1, \ldots, A_m$ of A is the *score* of the matrix A. Accordingly, the score of a row A_i of A is the binary number associated with A_i. Let $\mathcal{C}$ be the set of all matrices isomorphic to A. Matrix A is a *representative* of $\mathcal{C}$ if A has the maximal score among all matrices in $\mathcal{C}$. For $i = 1, \ldots, m$, we denote by $A(i)$ the row submatrix of A containing rows $1, \ldots, i$ of A.

Note that two matrices in $\mathcal{C}$ have the same score if and only if they are identical. The correctness of the algorithm given below relies on the following lemma:

Lemma 4.1 *Let A be an $(m \times n)$ $(0, 1)$-matrix. Then*

(i) if A is a representative then the score of the rows of A is nonincreasing.

(ii) A is a representative if and only if $A(i)$ is a representative, for $i = 1, \ldots, m$.

Proof. i) Suppose that there exists an index $i \in \{1, \ldots, m-1\}$ such that the score of A_i is strictly smaller than the score of A_{i+1} and let A' be the matrix obtained by exchanging the rows i and $i + 1$ in A. Then the score of A' is strictly greater than the score of A, implying that A is not a representative.

ii) As $A(m) = A$, it is immediate that if $A(i)$ is a representative for $i = 1, \ldots, m$ then A is a representative. To show the converse, suppose that there exists an index $i \in \{1, \ldots, m\}$ such that $A(i)$ is not a representative. Let $A'(i)$ be the representative of the set of matrices isomorphic to $A(i)$ and Π_l (resp. Π_c) be the permutation of the rows (resp. columns) of $A(i)$ used to obtain $A'(i)$ from $A(i)$. Let A'' be the matrix obtained from A by permuting its rows according to Π_l and its columns according to Π_c. Then $A''(i) = A'(i)$ and as the score of $A'(i)$ is strictly greater than the score of $A(i)$, the score of A'' is stricly greater than the score of A, showing that A is not a representative. ∎

The existence of a polynomial time algorithm for testing if a given matrix A is a representative is an open question and it is not even clear if this problem belongs to NP. Note that a polynomial time algorithm returning the representative of the isomorphism class of a given matrix could be used to decide in polynomial time if two given bipartite graphs are isomorphic, a problem whose complexity status is a long lasting open question [6][7]. For fixed r, testing if a matrix A with exactly r entries '1' per row and at most r '1's per column is a representative may possibly be solved in polynomial time by resorting to techniques developed to test the isomorphism of two graphs with bounded maximal degree [10]. As we deal only with small matrices, we used a exponential time algorithm for deciding if a matrix is a representative.

Let $r \geq 2$ be a fixed integer. For all $n \geq 3$, we define $T^{(n,r)}$ as the matrix whose rows are the $\left(\begin{array}{c} n \\ r \end{array} \right)$ distinct $(0,1)$-vectors of length n with exactly r '1's, ordered by decreasing score associated with each row. We denote by $T_i^{(n,r)}$ the ith row of this matrix.

The following algorithm is a simple backtracking algorithm that enumerates all representative (n,r)-regular matrices. At each forward step it adds a new row to the current matrix A and checks that the resulting matrix may be completed to obtain an (n,r)-regular matrix and, if this is the case, that the matrix is a representative. If the matrix fails one of these tests, then it is impossible to add rows to it to obtain a representative (n,r)-regular matrix and thus the algorithm performs a backtracking step. The latter consists in deleting the last row A_i of the current matrix A and selecting as new row to be added to A, the row following A_i in the matrix $T^{(n,r)}$. More formally:

Input: The dimension n of the matrices and the number r of '1's per row and per column.
Output: All representative (n,r)-regular matrices.

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1) $A(1) = T_1^{(n,r)}$; $i := 2$; last_choice[2] := 0;

2) While $i > 1$ do

        If $i = n+1$ or last_choice[$i$] = $\left( \begin{array}{c} n \\ r \end{array} \right)$ then
            $i := i - 1$;
        else
            last_choice[$i$] := last_choice[$i$] + 1;
            $A_{(i)} := T^{(n,r)}_{\text{last\_choice}\,[i]}$;
            last_choice[$i+1$] := last_choice[$i$] − 1;
            If it is impossible to complete $A(i)$ to obtain an $(n,r)$-regular matrix then
                $i := i - 1$;
            else
                If $A(i)$ is not a representative then
                    $i := i - 1$;
            $i := i + 1$;
            If $i = n + 1$ then output $A$;
```

The condition "While $i > 1$ do …" is satisfied until the algorithm attempts to replace the first row of A by the second row of $T^{(n,r)}$. As the only representative matrix with one row and exactly r '1's per row is the first row of $T^{(n,r)}$, this stopping criterion is correct. The validity of the remaining steps follows also from Lemma 4.1.

Note that this algorithm may be seen as a variant of the orderly generation technique of Read [16] (see also [11]), a reference brought to our attention after completion of this work.

5 Square Lehman matrices

For an $(n \times n)$ Lehman matrix A, we write bA the core of $b(A)$. Let r (resp. s) denotes the number of '1's in each row and column of A (resp. bA). By Lemma 2.6, the three integers n, r and s satisfy $1 \leq r \cdot s - n \leq \min\,(r-1, s-1)$. It follows that for $3 \leq n \leq 13$, the only feasible triplets (n, r, s) are:

$(3,2,2)$, $(5,2,3)$, $(5,3,2)$, $(7,2,4)$, $(7,3,3)$, $(7,4,2)$, $(8,3,3)$,
$(9,2,5)$, $(9,5,2)$, $(10,3,4)$, $(10,4,3)$, $(11,2,6)$, $(11,3,4)$, $(11,4,3)$,
$(11,6,2)$, $(13,2,7)$, $(13,3,5)$, $(13,4,4)$, $(13,5,3)$ $(13,7,2)$.

Running the algorithm described in the previous section with these parameters (n, r) yield the following square Lehman matrices:

(3,2): C_3^2.
(5,2): C_5^2.
(5,3): C_5^3.
(7,2): C_7^2.
(7,3): F_7.
(7,4): C_7^4.
(8,3): C_8^3, D_8.
(9,2): C_9^2.
(9,5): C_9^5.
(10,3): $\tau(K_5)$.
(10,4): $b\tau(K_5)$.
(11,2): C_{11}^2.
(11,3): C_{11}^3, $M_{11}^3(1)$, $M_{11}^3(2)$, $M_{11}^3(3)$.
(11,4): C_{11}^4, $M_{11}^4(1)$, $M_{11}^4(2)$, $L_{11}^4(3)$.
(11,6): C_{11}^6.
(13,2): C_{13}^2.
(13, 3): -.
(13, 4)*: $L_{13}^4(1)$.
(13, 5): -.
(13, 7): C_{13}^7.

Matrices C_n^k are the circulant matrices defined in Section 2 and all mni circulant matrices have been found prior to this work [1]. In the above list, all matrices except $L_{11}^4(3)$ and $L_{13}^4(1)$ are mni matrices. (The convention for the names of the noncirculant matrices is that 'M' is used for mni matrices and 'L' for Lehman matrices that are not mni matrices; all matrices whose names start with 'L' or 'M' have not appeared in the literature before). The numbering of these matrices reflects their blocking relation, i.e. $b(M_{11}^3(1)) = M_{11}^4(1)$ and $b(M_{11}^3(3)) = L_{11}^4(3)$. The star associated with parameters $(13, 4)$ indicates that the enumeration has not been completed. We thus have the list of all square mni matrices (nonisomorphic to a degenerate projective plane) with up to 12 columns.

Matrix F_7 is the matrix corresponding to the points and lines of the Fano plane, an mni matrix already given by Lehman [8]. Matrix D_8 is an mni matrix found by Ding [2] and is given in Section 7 together with all the new matrices cited in the paper. Matrix $\tau(K_5)$ is the matrix whose columns correspond to the edges of the complete graph on 5 nodes K_5 and whose rows are the incidence vectors of the triangles contained in K_5, an mni matrix given in [1].

For greater matrices, we only have partial results. We run the algorithm for $(14, 3)$, $(17, 3)$ and $(20, 3)$. There are two reasons behind the choice of these parameters: First, the algorithm is more sensitive to the value of r than of n and it would be terribly time consuming to run it with $n \geq 14$ and $r = 4$. (The enumeration with parameters $n = 14$, $r = 3$ took about 21 hours of cpu time on Silicon Graphics computers[1] and the very partial enumeration of the matrices with parameters (13, 4) took already more time; the enumeration for $n = 17$, $r = 3$ took about 257 days of cpu time). Secondly, only three cores of mni matrices with $r \cdot s - n > 1$ are known, namely F_7, $\tau(K_5)$ and $b(\tau(K_5))$. Thus, it is more likely to find new mni matrices for parameters satisfying $r \cdot s - n = 1$.

For (14, 3) we found 9 square mni matrices (the circulant matrix C_{14}^3 and 8 new matrices named $M_{14}^3(i)$ for $i = 1, \ldots, 8$) and 9 non-mni Lehman matrices (named $L_{14}^3(i)$ for $i = 9, \ldots, 17$). The cores of the respective blockers of these 18 matrices give 18 $(14, 5)$-Lehman matrices. Among these, 2 are mni matrices ($M_{14}^5(13) = bL_{14}^3(13)$ and $M_{14}^5(14) = bL_{14}^3(14)$).

For the parameters $(17, 3)$, we got 97 Lehman matrices and 4 of them are mni matrices (the circulant matrix C_{17}^3 and the matrices $M_{17}^3(i)$ for $i = 1, 2, 3$). A very partial enumeration for (20, 3) (probably much less than 1% of the full enumeration tree has been explored) yields 174 square Lehman matrices but no square mni matrix.

6 Nonsquare mni matrices

As mentioned in Section 2, we did not try to obtain all nonsquare mni matrices, but rather to list all possible cores of mni matrices. We know that all these cores are square Lehman matrices. The problem we face is thus to decide if, for a given square Lehman matrix A_1 with r '1's per row and per column, there exists a set of rows with at least $(r + 1)$ '1's that can be added to A_1 such that the resulting matrix A is a Lehman matrix with no dominating row and $Q(A)$ has a unique fractional extreme point, due to lemma 2.8.

The algorithm we used to find a completion of a Lehman matrix is a greedy heuristic. First, we construct a set of rows $\mathcal{R}$ with at least $(r + 1)$ '1's among which the rows to be added will be chosen. A sensible choice for $\mathcal{R}$ is the rows with at least $(r + 1)$ '1's of the blocker of the core of the blocker of A_1, since adding any subset of these rows to A_1 will yield a Lehman matrix. Then, at each step, add the row in $\mathcal{R}$ for which the resulting polytope $Q(A)$ has the minimum number of fractional extreme points. In case of tie, choose randomly one of the best

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[1] Iris Indigo MIPS R4000 100MHZ

rows. We used the code developed by Fukuda [3] to compute the extreme points and extreme rays of a polyhedron.

Despite its simplicity, this algorithm managed to complete 38 of the square nonmni Lehman matrices described in the previous section. More precisely, matrices $L^4_{11}(3)$, $L^3_{14}(i)$, for $i = 9, \ldots, 17$ and $L^3_{17}(i)$ for $i = 4 \ldots, 31$ have been completed. By taking the blockers of these 38 matrices, we get another 38 mni matrices with cores $M^3_{11}(3)$, $L^5_{14}(i)$ for $i = 9, \ldots, 17$ and $L^6_{17}(i)$ for $i = 4, \ldots, 31$.

Note that it is not known whether each Lehman matrix may be completed to an mni matrix or not. We believe that this is not the case, in particular due to the following construction: Let A be an $(n \times n)$-Lehman matrix with three '1's per row and per column and such that the last five rows of A are the characteristic vectors of the sets {n-4, n-3, n-2}, {n-3, n-2, n-1}, {n-2, n-1, n}, {1, n-1, n} and {1, 2, n}. Let A' be the $((n+3) \times (n+3))$-Lehman matrix obtained from A by adding 3 columns numbered (n+1), (n+2) and (n+3) and by replacing the last two rows of A by the characteristic vectors of the sets {n-1, n, n+1}, {n, n+1, n+2}, {$n+1$, n+2, n+3}, {1, n+2, n+3} and {1, 2, n+3}. It is then straightforward to show that A' is a Lehman matrix. As the last five rows of A' have the prescribed form, it is possible to apply repeatedly the same operation and thus obtaining an infinite familly of Lehman matrices. As a result, if it is always possible to add rows to any Lehman matrix to obtain an mni matrix, there are many infinite families of mni matrices and Conjecture 2.4 is disproved.

7 A catalog of Lehman and mni matrices

This section describes the matrices mentioned in the paper. A file named *mni_matrices.txt* containing these matrices can be retrieved by anonymous ftp at *ftp.epfl.ch* in the directory *pub/doc/dma*. As most of the cores of these matrices are isomorphic to "almost circulant" matrices, we describe them by indicating only the rows to modify in the circulant matrix C^r_n to obtain the matrix. Consider for example,

$$D_8 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix is obtained from C^3_8 by modifying the first and forth rows. We will note this by giving the name of the circulant matrix followed by a description of the rows to be modified, in the following format: *index*: (*array*), where *index* is the index of the row to be modified and *array* is the list of the positions of the entries '1' of the new row. For example, D_8 is given by:

$D_8 = C_8^3$ 1 : (1, 3, 5) 4 : (2, 4, 6)

A matrix with name starting with an 'M' is a square mni matrix. A matrix with name starting with an 'L' is a Lehman matrix that is not an mni matrix, but, in some cases, we managed to complete it into an mni matrix by adding a set of rows. For these nonsquare mni matrices, the additional rows are given immediately after the description of the core and are numbered with indices $(n+1)$, $(n+2)$, ..., where n is the number of columns of the matrix.

A) Matrices with 11 columns:

$M^3_{11}(1) = C^3_{11}$ 1 : (1, 3, 5) 4 : (2, 4, 6)
$M^3_{11}(2) = C^3_{11}$ 1 : (1, 2, 9) 4 : (5, 6, 9) 7 : (3, 7, 8) 9 : (4, 10, 11)
$M^3_{11}(3) = C^3_{11}$ 1 : (1, 2, 9) 6 : (3, 7, 8) 9 : (6, 10, 11)

$M^4_{11}(1) = C^4_{11}$ 1 : (2, 3, 4, 8) 5 : (1, 5, 6, 7)
$M^4_{11}(2) = C^4_{11}$ 1 : (2, 3, 4, 8) 3 : (4, 5, 6, 10) 5 : (1, 5, 6, 7) 7 : (3, 7, 8, 9)
$L^4_{11}(3) = C^4_{11}$ 1 : (1, 2, 3, 8) 2 : (3, 4, 5, 9) 6 : (2, 6, 7, 8) 8 : (4, 9, 10, 11) 12 : (4, 5, 7, 9, 10)

B) Matrix with 13 columns:

$L^4_{13}(1) = C^4_{13}$ 2 : (1, 5, 6, 7) 3 : (1, 8, 9, 10) 4 : (1, 11, 12, 13) 5 : (2, 5, 8, 11) 6 : (2, 6, 9, 12) 7 : (2, 7, 10, 13)
8 : (3, 5, 9, 13) 9 : (3, 6, 10, 11) 10 : (3, 7, 8, 12) 11 : (4, 5, 10, 12) 12 : (4, 6, 8, 13) 13 : (4, 7, 9, 11)

C) Matrices with 14 columns:

$M^3_{14}(1) = C^3_{14}$ 1 : (1, 3, 5) 4 : (2, 4, 6)
$M^3_{14}(2) = C^3_{14}$ 1 : (2, 3, 9) 7 : (1, 7, 8)
$M^3_{14}(3) = C^3_{14}$ 1 : (2, 3, 9) 7 : (1, 7, 8) 9 : (9, 10, 14) 14 : (1, 2, 11)
$M^3_{14}(4) = C^3_{14}$ 1 : (1, 2, 6) 4 : (4, 5, 9) 9 : (3, 10, 11)
$M^3_{14}(5) = C^3_{14}$ 1 : (1, 2, 6) 6 : (7, 8, 11) 9 : (3, 9, 10)
$M^3_{14}(6) = C^3_{14}$ 1 : (2, 3, 6) 4 : (1, 4, 5) 7 : (8, 9, 12) 10 : (7, 10, 11)
$M^3_{14}(7) = C^3_{14}$ 1 : (2, 3, 9) 4 : (4, 5, 12) 9 : (6, 10, 11) 10 : (1, 10, 11)
$M^3_{14}(8) = C^3_{14}$ 1 : (1, 2, 7) 3 : (3, 4, 9) 5 : (5, 6, 11) 7 : (7, 8, 13) 9 : (1, 9, 10) 11 : (3, 11, 12)
13 : (5, 13, 14)
$L^3_{14}(9) = C^3_{14}$ 1 : (2, 9, 12) 2 : (2, 3, 7) 7 : (4, 8, 9) 9 : (3, 10, 11) 10 : (1, 10, 11) 15 : (4, 5, 9, 12)
16 : (1, 4, 5, 9)
$L^3_{14}(10) = C^3_{14}$ 1 : (2, 9, 12) 6 : (3, 7, 8) 9 : (1, 10, 11) 10 : (6, 10, 11) 15 : (1, 3, 5, 7) 16 : (3, 5, 7, 12)
$L^3_{14}(11) = C^3_{14}$ 1 : (2, 6, 12) 4 : (4, 5, 9) 9 : (1, 10, 11) 10 : (3, 10, 11) 15 : (1, 4, 5, 14) 16 : (1, 4, 5, 6)
$L^3_{14}(12) = C^3_{14}$ 1 : (2, 3, 12) 4 : (5, 6, 9) 7 : (1, 7, 8) 10 : (4, 10, 11) 15 : (2, 10, 11, 12) 16 : (7, 10, 11, 12)
$L^3_{14}(13) = C^3_{14}$ 1 : (1, 2, 6) 6 : (3, 7, 8) 7 : (8, 9, 12) 10 : (7, 10, 11) 15 : (3, 5, 7, 11) 16 : (6, 7, 8, 10)
$L^3_{14}(14) = C^3_{14}$ 1 : (2, 3, 12) 4 : (5, 6, 9) 7 : (4, 7, 8) 10 : (1, 10, 11) 15 : (1, 6, 11, 13) 16 : (2, 7, 12, 14)
$L^3_{14}(15) = C^3_{14}$ 1 : (1, 2, 6) 4 : (5, 6, 9) 6 : (7, 8, 14) 7 : (7, 8, 12) 10 : (4, 10, 11) 12 : (3, 12, 13)
15 : (2, 6, 7, 8) 16 : (2, 3, 6, 7) 17 : (1, 5, 6, 13) 18 : (1, 4, 5, 6)
$L^3_{14}(16) = C^3_{14}$ 1 : (1, 2, 9) 2 : (2, 4, 12) 6 : (7, 8, 12) 9 : (3, 10, 11) 10 : (6, 10, 14) 12 : (3, 11, 13)
15 : (4, 5, 8, 12) 16 : (1, 8, 10, 14) 17 : (1, 2, 7, 12) 18 : (5, 6, 9, 10)
$L^3_{14}(17) = C^3_{14}$ 1 : (2, 6, 12) 6 : (7, 8, 11) 9 : (3, 9, 10) 10 : (1, 10, 11) 15 : (6, 10, 11, 12) 16 : (6, 9, 10, 12)
17 : (1, 2, 6, 10) 18 : (1, 5, 6, 10) 19 : (5, 6, 10, 12) 20 : (1, 6, 9, 10)

$M^5_{14}(13) = C^5_{14}$ 1 : (1, 2, 3, 4, 10) 3 : (4, 5, 6, 7, 12) 8 : (3, 8, 9, 10, 11) 10 : (5, 11, 12, 13, 14)
$M^5_{14}(14) = C^5_{14}$ 1 : (1, 2, 3, 4, 10) 7 : (2, 8, 9, 10, 11) 10 : (5, 11, 12, 13, 14) 12 : (1, 7, 12, 13, 14)

D) Matrices with 17 columns:

$M^3_{17}(1) = C^3_{17}$ $1 : (1,2,12)$ $12 : (3,13,14)$
$M^3_{17}(2) = C^3_{17}$ $1 : (1,2,12)$ $3 : (3,4,11)$ $11 : (5,12,13)$ $12 : (3,13,14)$
$M^3_{17}(3) = C^3_{17}$ $1 : (1,2,12)$ $2 : (3,4,10)$ $8 : (2,8,9)$ $12 : (3,13,14)$

$L^3_{17}(4) = C^3_{17}$ $1 : (1,2,6)$ $6 : (3,7,8)$ $8 : (8,9,11)$ $11 : (10,12,13)$ $18 : (1,9,13,14)$ $19 : (1,3,13,14)$
$20 : (3,4,13,14)$ $21 : (3,4,8,16)$
$L^3_{17}(5) = C^3_{17}$ $1 : (1,2,9)$ $4 : (5,6,9)$ $7 : (3,7,8)$ $9 : (4,10,11)$ $18 : (2,3,14,15)$ $19 : (2,10,14,15)$
$20 : (1,2,14,15)$ $21 : (3,5,14,15)$ $22 : (5,10,14,15)$ $23 : (5,6,14,15)$ $24 : (4,5,6,17)$
$L^3_{17}(6) = C^3_{17}$ $1 : (2,9,10)$ $6 : (3,7,8)$ $8 : (1,8,12)$ $10 : (2,6,12)$ $12 : (11,13,14)$ $17 : (1,10,17)$
$18 : (6,10,11,12)$ $19 : (6,10,11,17)$ $20 : (3,4,11,16)$ $21 : (6,11,12,16)$ $22 : (9,11,16,17)$ $23 : (3,11,16,17)$
$24 : (6,11,16,17)$ $25 : (10,11,13,17)$ $26 : (11,13,16,17)$ $27 : (3,10,11,17)$ $28 : (3,7,10,11)$ $29 : (3,10,11,12)$
$30 : (3,4,10,11)$ $31 : (1,3,4,8)$ $32 : (3,4,8,11)$
$L^3_{17}(7) = C^3_{17}$ $1 : (1,2,9)$ $6 : (3,7,8)$ $9 : (6,10,11)$ $18 : (1,4,5,17)$ $19 : (1,5,7,17)$ $20 : (1,7,8,17)$
$21 : (6,7,8,11)$ $22 : (6,7,8,17)$ $23 : (3,5,7,12)$ $24 : (3,5,7,15)$ $25 : (1,3,5,7)$
$L^3_{17}(8) = C^3_{17}$ $1 : (1,2,12)$ $6 : (7,8,11)$ $9 : (6,9,10)$ $12 : (3,13,14)$ $18 : (3,7,14,16)$ $19 : (3,10,14,16)$
$20 : (5,6,10,17)$ $21 : (3,4,11,16)$ $22 : (3,4,14,16)$ $23 : (3,4,9,16)$ $24 : (1,3,14,16)$ $25 : (5,6,10,12)$
$26 : (1,2,3,14)$ $27 : (5,6,10,14)$ $28 : (4,11,12,16)$ $29 : (1,11,12,16)$ $30 : (11,12,15,16)$ $31 : (7,11,12,16)$
$L^3_{17}(9) = C^3_{17}$ $1 : (1,2,6)$ $6 : (7,8,11)$ $9 : (3,9,10)$ $18 : (1,2,8,9)$ $19 : (1,2,8,14)$ $20 : (1,9,10,17)$
$21 : (6,9,10,12)$
$L^3_{17}(10) = C^3_{17}$ $1 : (1,2,10)$ $3 : (3,4,14)$ $5 : (5,6,15)$ $10 : (3,11,12)$ $13 : (7,13,14)$ $14 : (5,15,16)$
$18 : (1,13,14,17)$ $19 : (1,3,14,17)$ $20 : (1,3,10,14)$ $21 : (1,10,13,14)$ $22 : (1,5,6,10)$ $23 : (9,10,13,14)$
$24 : (1,5,6,17)$ $25 : (3,12,14,15)$ $26 : (3,8,12,14)$ $27 : (1,3,12,14)$ $28 : (4,5,13,14)$ $29 : (4,10,13,14)$
$L^3_{17}(11) = C^3_{17}$ $1 : (1,2,6)$ $4 : (4,5,12)$ $7 : (8,9,12)$ $10 : (7,10,11)$ $12 : (3,13,14)$ $18 : (2,9,14,15)$
$19 : (8,9,14,15)$ $20 : (2,4,14,15)$ $21 : (2,3,6,7)$ $22 : (2,7,14,15)$ $23 : (2,6,7,11)$ $24 : (2,9,11,12)$
$25 : (3,4,8,9)$ $26 : (3,4,8,13)$ $27 : (3,4,8,16)$ $28 : (3,4,11,16)$ $29 : (1,2,14,15)$ $30 : (5,9,11,12)$
$31 : (9,11,12,16)$ $32 : (3,4,14,16)$ $33 : (6,7,11,16)$
$L^3_{17}(12) = C^3_{17}$ $1 : (1,2,9)$ $4 : (4,5,12)$ $9 : (6,10,11)$ $12 : (3,13,14)$ $18 : (7,8,11,12)$ $19 : (2,7,11,12)$
$20 : (7,11,12,16)$ $21 : (2,4,11,12)$ $22 : (2,3,6,7)$ $23 : (2,6,7,11)$ $24 : (6,7,11,16)$ $25 : (4,5,6,10)$
$26 : (1,4,5,6)$ $27 : (4,5,6,15)$
$L^3_{17}(13) = C^3_{17}$ $1 : (1,2,7)$ $5 : (5,7,11)$ $6 : (6,8,12)$ $11 : (6,12,13)$ $12 : (3,13,14)$ $18 : (5,6,7,8)$
$19 : (1,6,13,14)$ $20 : (5,6,13,14)$ $21 : (5,6,8,14)$ $22 : (4,6,8,9)$ $23 : (1,2,4,6)$ $24 : (6,8,9,14)$
$L^3_{17}(14) = C^3_{17}$ $1 : (1,2,6)$ $6 : (3,7,8)$ $8 : (9,10,13)$ $11 : (8,11,12)$ $18 : (1,6,7,8)$ $19 : (5,6,11,12)$
$20 : (6,7,8,12)$ $21 : (5,11,12,14)$
$L^3_{17}(15) = C^3_{17}$ $1 : (1,2,9)$ $6 : (3,7,8)$ $7 : (7,8,12)$ $12 : (6,13,14)$ $18 : (6,8,10,14)$ $19 : (6,7,8,14)$
$20 : (3,8,10,17)$ $21 : (3,4,8,10)$ $22 : (1,2,6,14)$ $23 : (1,2,14,15)$ $24 : (3,8,10,14)$ $25 : (8,10,12,17)$
$26 : (4,8,10,12)$ $27 : (8,10,12,14)$ $28 : (1,7,8,9)$ $29 : (6,7,8,9)$
$L^3_{17}(16) = C^3_{17}$ $1 : (1,2,9)$ $6 : (3,7,8)$ $9 : (10,11,14)$ $12 : (6,12,13)$ $18 : (7,8,15,17)$ $19 : (7,8,14,15)$
$20 : (7,8,11,17)$ $21 : (6,7,8,12)$ $22 : (6,7,8,17)$ $23 : (8,10,14,15)$ $24 : (5,10,14,15)$
$L^3_{17}(17) = C^3_{17}$ $1 : (2,3,6)$ $4 : (4,5,12)$ $10 : (1,10,11)$ $18 : (7,11,13,15)$ $19 : (3,11,13,15)$ $20 : (3,8,9,16)$
$21 : (10,11,13,15)$ $22 : (6,10,11,15)$ $23 : (11,13,15,17)$ $24 : (1,4,5,17)$ $25 : (2,3,8,9)$ $26 : (1,11,13,17)$
$27 : (3,8,9,13)$ $28 : (4,5,15,17)$ $29 : (4,10,11,15)$ $30 : (4,5,10,15)$ $31 : (4,5,14,15)$ $32 : (4,5,7,15)$
$L^3_{17}(18) = C^3_{17}$ $1 : (1,2,6)$ $4 : (4,5,9)$ $9 : (10,11,14)$ $12 : (3,12,13)$ $18 : (1,9,12,13)$ $19 : (5,9,12,13)$
$20 : (4,5,11,17)$ $21 : (4,5,11,12)$ $22 : (4,5,6,11)$ $23 : (9,12,13,15)$
$L^3_{17}(19) = C^3_{17}$ $1 : (2,3,7)$ $2 : (2,3,12)$ $7 : (1,4,8)$ $12 : (9,13,14)$ $18 : (2,9,10,17)$ $19 : (5,9,10,17)$
$20 : (3,5,9,10)$ $21 : (5,6,9,10)$ $22 : (2,6,7,17)$ $23 : (2,6,7,14)$ $24 : (2,9,10,14)$ $25 : (5,9,10,14)$
$26 : (6,7,11,16)$ $27 : (6,7,10,11)$ $28 : (2,6,7,11)$
$L^3_{17}(20) = C^3_{17}$ $1 : (1,2,12)$ $3 : (3,4,13)$ $7 : (7,8,15)$ $11 : (11,12,16)$ $12 : (9,13,14)$ $14 : (5,14,15)$
$15 : (3,16,17)$ $18 : (5,6,9,14)$ $19 : (2,7,11,12)$ $20 : (2,11,12,14)$ $21 : (3,4,5,10)$ $22 : (3,4,5,15)$
$23 : (3,4,5,16)$
$L^3_{17}(21) = C^3_{17}$ $1 : (2,3,15)$ $4 : (4,5,12)$ $5 : (5,6,10)$ $10 : (7,11,12)$ $12 : (6,13,14)$ $13 : (1,13,14)$
$18 : (1,4,5,17)$ $19 : (1,5,7,17)$ $20 : (1,11,13,17)$ $21 : (1,5,13,17)$ $22 : (1,3,7,8)$ $23 : (1,7,8,17)$
$24 : (4,7,8,12)$ $25 : (1,7,8,12)$
$L^3_{17}(22) = C^3_{17}$ $1 : (2,3,15)$ $4 : (4,5,12)$ $9 : (6,10,11)$ $12 : (9,13,14)$ $13 : (1,13,14)$ $18 : (1,3,10,11)$
$19 : (1,10,11,17)$ $20 : (3,9,10,11)$ $21 : (1,11,13,17)$ $22 : (1,8,13,17)$
$L^3_{17}(23) = C^3_{17}$ $1 : (2,3,15)$ $4 : (4,5,12)$ $6 : (6,7,11)$ $11 : (8,12,13)$ $12 : (1,13,14)$ $13 : (6,13,14)$
$18 : (1,11,13,17)$ $19 : (1,4,5,17)$ $20 : (1,5,13,17)$ $21 : (1,3,10,11)$ $22 : (1,10,11,17)$ $23 : (3,6,10,11)$
$L^3_{17}(24) = C^3_{17}$ $1 : (1,2,6)$ $6 : (3,7,8)$ $8 : (8,9,13)$ $13 : (10,14,15)$ $18 : (1,2,3,8)$ $19 : (1,2,3,11)$
$20 : (3,4,16,17)$ $21 : (4,12,16,17)$
$L^3_{17}(25) = C^3_{17}$ $1 : (1,2,6)$ $6 : (7,8,14)$ $9 : (3,9,10)$ $14 : (11,15,16)$ $18 : (5,7,8,11)$ $19 : (6,9,10,12)$
$L^3_{17}(26) = C^3_{17}$ $1 : (1,2,6)$ $6 : (3,7,11)$ $11 : (8,12,13)$ $18 : (1,2,14,15)$ $19 : (1,2,9,14)$ $20 : (1,2,3,14)$
$21 : (7,9,11,14)$ $22 : (7,9,11,17)$ $23 : (3,7,8,13)$ $24 : (1,6,7,11)$ $25 : (3,5,7,8)$ $26 : (6,7,9,11)$

$L^3_{17}(27) = C^3_{17}$	1 : (1, 2, 12)	4 : (4, 5, 9)	9 : (6, 10, 11)	12 : (3, 13, 14)	18 : (4, 5, 6, 11)	19 : (1, 4, 5, 6)
	20 : (4, 9, 10, 11)	21 : (9, 10, 11, 14)				
$L^3_{17}(28) = C^3_{17}$	1 : (2, 3, 15)	2 : (2, 3, 10)	7 : (4, 8, 9)	10 : (1, 11, 12)	13 : (7, 13, 14)	18 : (4, 5, 9, 11)
	19 : (6, 11, 12, 15)	20 : (9, 11, 12, 15)	21 : (2, 3, 4, 13)	22 : (2, 3, 4, 8)	23 : (1, 2, 3, 4)	
$L^3_{17}(29) = C^3_{17}$	1 : (1, 2, 15)	4 : (5, 6, 12)	7 : (4, 7, 8)	12 : (9, 13, 14)	15 : (3, 16, 17)	18 : (3, 4, 14, 16)
	19 : (3, 4, 11, 16)	20 : (7, 8, 10, 17)				
$L^3_{17}(30) = C^3_{17}$	1 : (2, 9, 15)	3 : (3, 4, 10)	7 : (7, 8, 12)	8 : (8, 9, 13)	11 : (5, 11, 12)	12 : (3, 13, 14)
	13 : (1, 13, 14)	18 : (7, 10, 12, 15)	19 : (2, 11, 12, 15)	20 : (2, 7, 12, 15)	21 : (1, 2, 11, 12)	22 : (2, 3, 11, 12)
	23 : (8, 11, 12, 13)	24 : (2, 8, 11, 12)	25 : (9, 13, 14, 15)	26 : (9, 10, 13, 15)	27 : (5, 7, 12, 15)	28 : (3, 4, 5, 12)
	29 : (5, 9, 13, 15)	30 : (9, 10, 15, 16)	31 : (5, 9, 11, 13)	32 : (5, 9, 15, 16)		
$L^3_{17}(31) = C^3_{17}$	1 : (1, 2, 15)	4 : (5, 6, 10)	5 : (5, 6, 15)	10 : (4, 11, 12)	13 : (7, 13, 14)	15 : (3, 16, 17)
	18 : (1, 2, 6, 10)	19 : (1, 2, 10, 13)	20 : (1, 2, 4, 10)	21 : (2, 9, 10, 14)	22 : (1, 2, 9, 10)	23 : (5, 9, 10, 14)
$L^3_{17}(32) = C^3_{17}$	1 : (1, 10, 14)	6 : (3, 7, 8)	9 : (2, 6, 17)	14 : (9, 15, 16)	15 : (11, 15, 16)	
$L^3_{17}(33) = C^3_{17}$	1 : (3, 7, 8)	6 : (1, 2, 6)	8 : (8, 9, 13)	11 : (11, 12, 16)	16 : (1, 10, 17)	
$L^3_{17}(34) = C^3_{17}$	1 : (7, 8, 12)	6 : (1, 2, 6)	7 : (3, 7, 8)	12 : (9, 13, 17)	17 : (1, 2, 14)	
$L^3_{17}(35) = C^3_{17}$	1 : (1, 13, 17)	6 : (2, 6, 10)	8 : (3, 7, 9)	11 : (8, 12, 16)	16 : (1, 2, 8)	17 : (1, 11, 17)
$L^3_{17}(36) = C^3_{17}$	1 : (11, 13, 17)	6 : (2, 6, 10)	8 : (3, 7, 9)	11 : (1, 8, 12)	12 : (1, 12, 14)	17 : (2, 8, 13)
$L^3_{17}(37) = C^3_{17}$	1 : (1, 2, 14)	6 : (2, 6, 8)	7 : (3, 7, 9)	9 : (1, 7, 9)	14 : (11, 15, 17)	17 : (8, 10, 16)
$L^3_{17}(38) = C^3_{17}$	1 : (2, 6, 17)	6 : (1, 3, 7)	7 : (1, 7, 8)	12 : (9, 13, 14)	15 : (2, 8, 15)	17 : (12, 16, 17)
$L^3_{17}(39) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 17)	9 : (1, 2, 9)	14 : (11, 15, 16)	15 : (8, 15, 16)	17 : (10, 14, 17)
$L^3_{17}(40) = C^3_{17}$	1 : (1, 8, 11)	6 : (3, 7, 8)	8 : (2, 9, 10)	9 : (9, 10, 14)	12 : (12, 13, 17)	17 : (1, 2, 6)
$L^3_{17}(41) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (2, 9, 13)	13 : (1, 14, 15)	16 : (10, 16, 17)	17 : (1, 8, 17)
$L^3_{17}(42) = C^3_{17}$	1 : (2, 6, 15)	6 : (3, 7, 8)	8 : (2, 9, 13)	13 : (1, 10, 14)	17 : (1, 8, 17)	
$L^3_{17}(43) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (2, 9, 10)	11 : (8, 12, 16)	16 : (1, 11, 17)	17 : (1, 13, 17)
$L^3_{17}(44) = C^3_{17}$	1 : (2, 11, 15)	5 : (5, 6, 16)	9 : (2, 9, 15)	14 : (1, 10, 14)	15 : (3, 16, 17)	17 : (1, 7, 17)
$L^3_{17}(45) = C^3_{17}$	1 : (2, 6, 15)	6 : (3, 7, 8)	8 : (2, 9, 10)	10 : (1, 10, 11)	15 : (12, 16, 17)	17 : (1, 8, 17)
$L^3_{17}(46) = C^3_{17}$	1 : (1, 11, 14)	6 : (3, 7, 8)	8 : (2, 9, 10)	11 : (12, 13, 17)	12 : (8, 12, 13)	17 : (1, 2, 6)
$L^3_{17}(47) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (9, 10, 16)	11 : (2, 11, 12)	16 : (1, 8, 17)	17 : (1, 13, 17)
$L^3_{17}(48) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (9, 10, 16)	11 : (2, 12, 13)	14 : (11, 14, 15)	17 : (1, 8, 17)
$L^3_{17}(49) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (9, 13, 16)	13 : (10, 14, 15)	14 : (2, 14, 15)	17 : (1, 8, 17)
$L^3_{17}(50) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (9, 10, 13)	11 : (11, 12, 16)	14 : (2, 14, 15)	17 : (1, 8, 17)
$L^3_{17}(51) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	8 : (9, 10, 13)	11 : (2, 11, 12)	12 : (12, 13, 17)	17 : (1, 8, 14)
$L^3_{17}(52) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 8)	9 : (9, 10, 14)	14 : (2, 15, 16)	17 : (1, 11, 17)	
$L^3_{17}(53) = C^3_{17}$	1 : (2, 6, 17)	6 : (3, 7, 8)	9 : (1, 10, 11)	12 : (9, 12, 13)	13 : (2, 14, 15)	17 : (1, 13, 14)
$L^3_{17}(54) = C^3_{17}$	1 : (2, 6, 15)	6 : (3, 7, 8)	8 : (9, 10, 14)	10 : (1, 11, 12)	12 : (10, 13, 14)	13 : (2, 13, 14)
	14 : (12, 15, 16)	17 : (1, 8, 17)				
$L^3_{17}(55) = C^3_{17}$	1 : (2, 6, 16)	6 : (1, 3, 7)	10 : (11, 12, 17)	12 : (2, 13, 14)	15 : (8, 15, 16)	16 : (1, 12, 17)
	17 : (1, 10, 17)					
$L^3_{17}(56) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 11)	11 : (12, 13, 16)	14 : (2, 14, 15)	17 : (1, 8, 17)	
$L^3_{17}(57) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 11)	9 : (10, 11, 14)	11 : (2, 12, 13)	12 : (12, 13, 17)	15 : (9, 15, 16)
	17 : (1, 8, 17)					
$L^3_{17}(58) = C^3_{17}$	1 : (1, 2, 8)	6 : (2, 6, 13)	7 : (3, 7, 9)	8 : (7, 12, 16)	9 : (8, 9, 10)	10 : (9, 10, 11)
	11 : (10, 11, 12)	14 : (8, 14, 15)	17 : (1, 11, 17)			
$L^3_{17}(59) = C^3_{17}$	1 : (1, 2, 6)	6 : (3, 7, 11)	9 : (9, 10, 17)	11 : (2, 12, 13)	14 : (11, 15, 16)	17 : (1, 8, 14)
$L^3_{17}(60) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 17)	9 : (3, 9, 10)	14 : (11, 15, 16)	17 : (1, 2, 14)	
$L^3_{17}(61) = C^3_{17}$	1 : (2, 6, 12)	6 : (7, 8, 11)	9 : (3, 9, 10)	10 : (1, 10, 11)	14 : (2, 15, 16)	17 : (1, 14, 17)
$L^3_{17}(62) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 14)	9 : (3, 9, 10)	14 : (2, 15, 16)	17 : (1, 11, 17)	
$L^3_{17}(63) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 17)	8 : (9, 10, 16)	10 : (10, 11, 15)	14 : (8, 14, 15)	15 : (3, 12, 16)
$L^3_{17}(64) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 15)	7 : (7, 8, 17)	11 : (12, 13, 16)	14 : (11, 14, 15)	15 : (3, 9, 16)
$L^3_{17}(65) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 17)	9 : (3, 10, 11)	12 : (9, 13, 14)	15 : (12, 15, 16)	
$L^3_{17}(66) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 17)	9 : (10, 11, 14)	12 : (3, 12, 13)	15 : (9, 15, 16)	
$L^3_{17}(67) = C^3_{17}$	1 : (1, 2, 6)	6 : (7, 8, 14)	9 : (10, 11, 17)	12 : (9, 12, 13)	15 : (3, 15, 16)	
$L^3_{17}(68) = C^3_{17}$	1 : (2, 6, 10)	6 : (7, 8, 12)	8 : (2, 10, 15)	10 : (9, 11, 16)	13 : (1, 13, 14)	16 : (1, 8, 17)
	17 : (1, 3, 17)					
$L^3_{17}(69) = C^3_{17}$	1 : (2, 3, 9)	7 : (7, 8, 12)	10 : (11, 13, 17)	12 : (1, 2, 12)	17 : (1, 10, 14)	
$L^3_{17}(70) = C^3_{17}$	1 : (2, 3, 9)	7 : (7, 8, 12)	10 : (1, 2, 12)	12 : (11, 13, 14)	14 : (10, 15, 16)	17 : (1, 14, 17)
$L^3_{17}(71) = C^3_{17}$	1 : (1, 2, 11)	9 : (10, 12, 14)	10 : (3, 10, 17)	13 : (11, 13, 15)	17 : (1, 2, 9)	
$L^3_{17}(72) = C^3_{17}$	1 : (1, 2, 9)	9 : (3, 10, 14)	14 : (11, 15, 16)			
$L^3_{17}(73) = C^3_{17}$	1 : (1, 2, 9)	9 : (3, 10, 14)	14 : (2, 15, 16)	17 : (1, 11, 17)		
$L^3_{17}(74) = C^3_{17}$	1 : (1, 2, 11)	9 : (3, 10, 12)	10 : (11, 12, 13)	11 : (12, 13, 14)	12 : (13, 14, 15)	13 : (10, 14, 15)
	14 : (11, 15, 16)	15 : (9, 16, 17)				
$L^3_{17}(75) = C^3_{17}$	1 : (1, 2, 12)	9 : (3, 10, 11)	12 : (13, 14, 17)	15 : (9, 15, 16)		
$L^3_{17}(76) = C^3_{17}$	1 : (1, 2, 9)	7 : (7, 8, 12)	12 : (3, 13, 14)	14 : (2, 15, 16)	17 : (1, 14, 17)	
$L^3_{17}(77) = C^3_{17}$	1 : (1, 2, 11)	7 : (7, 8, 17)	9 : (9, 10, 16)	13 : (1, 14, 15)	14 : (3, 14, 15)	16 : (13, 16, 17)
	17 : (1, 2, 9)					

$L^3_{17}(78) = C^3_{17}$	$1 : \ (1,2,12)$	$8 : \ (9,10,16)$	$11 : \ (8,12,13)$	$12 : \ (13,14,17)$	$14 : \ (11,14,15)$	$15 : \ (3,15,16)$
$L^3_{17}(79) = C^3_{17}$	$1 : \ (2,9,15)$	$7 : \ (7,8,12)$	$10 : \ (1,10,11)$	$12 : \ (3,13,14)$	$15 : \ (12,16,17)$	
$L^3_{17}(80) = C^3_{17}$	$1 : \ (1,3,5)$	$4 : \ (2,4,6)$	$10 : \ (10,12,14)$	$13 : \ (11,13,15)$		
$L^3_{17}(81) = C^3_{17}$	$1 : \ (1,2,15)$	$7 : \ (8,9,13)$	$8 : \ (8,9,15)$	$13 : \ (7,10,14)$	$15 : \ (3,16,17)$	
$L^3_{17}(82) = C^3_{17}$	$1 : \ (1,2,15)$	$7 : \ (8,9,15)$	$10 : \ (7,11,12)$	$13 : \ (10,13,14)$	$15 : \ (3,16,17)$	
$L^3_{17}(83) = C^3_{17}$	$1 : \ (1,2,15)$	$7 : \ (8,12,15)$	$12 : \ (9,13,14)$	$13 : \ (7,13,14)$	$15 : \ (3,16,17)$	
$L^3_{17}(84) = C^3_{17}$	$1 : \ (2,3,15)$	$6 : \ (7,8,11)$	$9 : \ (9,10,14)$	$12 : \ (6,12,13)$	$13 : \ (1,13,14)$	
$L^3_{17}(85) = C^3_{17}$	$1 : \ (1,2,14)$	$7 : \ (1,8,9)$	$8 : \ (8,9,16)$	$10 : \ (7,10,11)$	$12 : \ (3,12,13)$	$14 : \ (10,14,15)$
	$16 : \ (12,16,17)$					
$L^3_{17}(86) = C^3_{17}$	$1 : \ (1,2,12)$	$6 : \ (6,7,17)$	$7 : \ (3,8,9)$	$10 : \ (11,12,15)$	$12 : \ (7,13,14)$	$13 : \ (8,13,14)$
	$17 : \ (1,2,10)$					
$L^3_{17}(87) = C^3_{17}$	$1 : \ (1,2,7)$	$5 : \ (5,6,14)$	$6 : \ (2,6,7)$	$7 : \ (3,8,9)$	$11 : \ (8,12,13)$	$14 : \ (7,15,16)$
	$17 : \ (1,11,17)$					
$L^3_{17}(88) = C^3_{17}$	$1 : \ (2,3,14)$	$5 : \ (1,6,7)$	$8 : \ (2,9,10)$	$9 : \ (10,11,17)$	$11 : \ (8,11,12)$	$12 : \ (1,12,13)$
	$14 : \ (5,13,14)$	$17 : \ (9,15,16)$				
$L^3_{17}(89) = C^3_{17}$	$1 : \ (2,6,7)$	$5 : \ (5,6,13)$	$6 : \ (7,8,12)$	$8 : \ (1,8,9)$	$10 : \ (10,11,17)$	$13 : \ (10,14,15)$
	$15 : \ (3,15,16)$					
$L^3_{17}(90) = C^3_{17}$	$1 : \ (2,6,10)$	$6 : \ (7,9,16)$	$8 : \ (1,8,14)$	$10 : \ (10,11,17)$	$12 : \ (2,12,13)$	$14 : \ (3,14,15)$
	$17 : \ (1,8,12)$					
$L^3_{17}(91) = C^3_{17}$	$1 : \ (1,8,12)$	$6 : \ (2,6,14)$	$8 : \ (7,9,13)$	$10 : \ (3,10,11)$	$11 : \ (8,12,16)$	$14 : \ (11,15,16)$
	$16 : \ (1,10,17)$					
$L^3_{17}(92) = C^3_{17}$	$1 : \ (2,6,17)$	$6 : \ (1,7,8)$	$9 : \ (3,10,11)$	$11 : \ (2,12,13)$	$14 : \ (11,15,16)$	$15 : \ (9,15,16)$
	$17 : \ (1,14,17)$					
$L^3_{17}(93) = C^3_{17}$	$1 : \ (1,2,6)$	$6 : \ (3,8,16)$	$7 : \ (7,9,17)$	$11 : \ (7,12,13)$	$14 : \ (11,14,15)$	$15 : \ (8,15,16)$
$L^3_{17}(94) = C^3_{17}$	$1 : \ (1,2,6)$	$6 : \ (7,8,14)$	$7 : \ (1,8,9)$	$10 : \ (7,11,12)$	$12 : \ (12,13,17)$	$15 : \ (3,15,16)$
	$16 : \ (10,16,17)$					
$L^3_{17}(95) = C^3_{17}$	$1 : \ (1,9,13)$	$6 : \ (7,8,14)$	$9 : \ (3,10,11)$	$12 : \ (2,12,17)$	$15 : \ (6,15,16)$	
$L^3_{17}(96) = C^3_{17}$	$1 : \ (1,2,12)$	$6 : \ (7,8,14)$	$9 : \ (6,10,11)$	$12 : \ (9,13,17)$	$15 : \ (3,15,16)$	

Acknowledgements

We would like to thank Gérard Cornuéjols for several discussions and helpful suggestions.

References

[1] Cornuéjols G., Novick B., "Ideal 0, 1 matrices", *J. Comb. Theory Ser. B* 60 (1994), 145-157.

[2] Ding G., "Clutters with $\tau_2 = 2 \cdot \tau$", *Discr. Math.* 115 (1993), 141-152.

[3] Fukuda K., "cdd.c", a C code for enumerating vertices and extreme rays, available by anonymous ftp from *ifor13.ethz.ch*, directory *pub/fukuda/cdd*.

[4] Fukuda K., private communication (1995).

[5] Fukuda K., Matsui T., "Finding all the perfect matchings in bipartite graphs", *Appl. Math. Lett.* 7 (1994), 15-18.

[6] Garey M.R., Johnson D.S., *Computers and Intractability, a Guide to the Theory of NP-Completeness*, Freeman (1979).

[7] Johnson D.S., "The NP-completness column, an ongoing guide", *J. Algorithms* 3 (1982), 288-300.

[8] Lehman A., "On the width-length inequality", mimeographic notes (1965), *Math. Progr.* 17 (1979), 403-417.

[9] Lehman A., "On the width-length inequality and degenerate projective planes", in *Polyhedral Combinatorics*, W. Cook and P.D. Seymour Eds., *DIMACS series in Discrete Mathematics and Computer Science*, vol. 1, Amer. Math. Soc. (1990), 101-105.

[10] Luks E.M., "Isomorphism of bounded valence can be tested in polynomial time", *Proc. FOCS* 21 (1980), 42-49.

[11] McKay B.D., "Isomorphism-Free Exhaustive Generation", Research report (1996), Australian National University, Canberra.

[12] Nemhauser G., Wolsey L., *Integer and Combinatorial Optimization*, Wiley (1988).

[13] Novick B., "Ideal 0,1 matrices", Ph.D. dissertation (1990), Carnegie Mellon University, Pittsburgh, PA.

[14] Padberg M.W., "Lehman's forbidden minor characterization of ideal 0-1 matrices", *Discr. Math.* 111 (1993), 409-420.

[15] Qi L., "On the set covering polytope and the MFMC-clutter", School of Mathematics, University of South Wales (1989).

[16] Read R.C., "A Survey of Graph Generation Techniques", in *Combinatorial Mathematics VIII, Lecture Notes in Mathematics* 884, 77-89, Springer (1981).

[17] Seymour P.D., "The matroids with the max-flow min-cut property", *J. Comb. Theory Ser. B* 23 (1977), 189-222.

[18] Seymour P.D., "On Lehman's width-length characterization", in *Polyhedral Combinatorics*, W. Cook and P.D. Seymour Eds., *DIMACS series in Discrete Mathematics and Computer Science*, vol. 1, Amer. Math. Soc. (1990), 107-117.

[19] L.G. Valiant, " The complexity of enumeration and reliability problems," *SIAM J. Comput.* 8 (1979), 410-421.