MIN-UP/MIN-DOWN POLYTOPES

Jon LEE[1] Janny LEUNG[2] François MARGOT[3]

February 2003

Abstract

In power generation and other production settings, technological constraints force restrictions on the number of time-periods that a machine must stay up once activated, and stay down once deactivated. We characterize the polyhedral structure of a model representing these restrictions. We also describe a cutting-plane method for solving integer programs involving such min-up and min-down times for machines. Finally, we demonstrate how the polytope of our study generalizes the well-known cross polytope (i.e., generalized octahedron).

Keywords: mixed integer programming, cutting plane, facet, separation, cross polytope, generalized octahedron, power generation, unit commitment

[1]Department of Mathematical Sciences, IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, U.S.A. Email: jonlee@us.ibm.com .
[2]Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong. Email: janny@se.cuhk.edu.hk .
[3]Department of Mathematics, University of Kentucky, 715 POT, Lexington, KY 40506-0027, U.S.A. Email: fmargot@ms.uky.edu .

Introduction

We assume familiarity with mixed integer programming (see (Nemhauser and Wolsey, 1999), for example). We consider a discrete planning problem with a T-period planning horizon. We have a set of N machines that produce a single perishable commodity in arbitrary (nonnegative) amounts. The aggregate demand for this commodity at time t is d_t. At each discrete time point t in $[1, T]$, machine i ($1 \leq i \leq N$) is *up* (i.e., operating) or *down* (i.e., not operating). The 0/1 indicator variable x_t^i indicates whether machine i is up at time t. The amount of the commodity produced by machine i in time period t is z_t^i. So we have the constraint

$$\sum_{i=1}^{N} z_t^i \geq d_t \text{ , for } t \text{ in } [1, T] \text{ .} \tag{1}$$

The amount z_t^i is required to be between q^i and Q^i whenever machine i is operating. So, we have the constraints:

$$q^i x_t^i \leq z_t^i \leq Q^i x_t^i \text{ , for } t \text{ in } [1, T] \text{ , and } i = 1, \ldots, N \text{ .} \tag{2}$$

An arbitrary linear objective in these variables is usually employed.

This and similar models have been called *unit-commitment problems* (see (Birge, Takriti and Long, 1995), for example). Recently, unit-commitment problems, models and solution methods have been the subject of intense inquiry, engendering much interest as evidenced by the recent DIMACS/EPRI Workshop on the Next Generation of Unit Commitment Models (27 – 28 September 1999, DIMACS Center, Rutgers University, Piscataway, NJ); see (Rothkopf, Hobbs, O'Neill and Chao, 2001).

One variation to the basic model involves restrictions on how quickly the state of a machine can change. Frequent changes between up and down have several adverse consequences including (i) increased operator stress, (ii) diminished generator life, and (iii) increased emission of pollutants during transient periods (see (Birge et al., 1995)). Let L^i and l^i be positive integers, for $i = 1, \ldots, N$. Further restrictions, which are the focus of our study, are that when machine i switches from down to up, it must then be up for at least L^i periods (or at least until the end of the planning horizon), and when machine i switches from up to down, it must then be down for at least l^i periods (or at least until the end of the planning horizon). Such restrictions are quite practical and are included in commercial tools (e.g., **PLEXOS for Power Systems**©, Version 4.5 — an electricity market simulation tool — has a unit-commitment model and solver which accepts restrictions of this type; see `http://www.plexos.info/`).

(Takriti, Krasenbrink and Wu, 2000), in their study of a unit commitment problem, formulate the min-up restrictions as:

$$x_t^i - x_{t-1}^i \le x_\tau^i \text{ , for } 2 \le t < \tau \le \min\{t + L^i - 1, T\} \ . \tag{3}$$

The left-hand side of this inequality is 1 exactly when machine i switches from down in period $t-1$ to up in period t. The inequalities for $\tau = t+1, \ldots, t+L^i-1$ then force the machine to be up for the $L^i - 1$ periods that immediately follow period t. Similarly, they formulate the min-down restrictions as:

$$x_{t-1}^i - x_t^i \le 1 - x_\tau^i \text{ , for } 2 \le t < \tau \le \min\{t + l^i - 1, T\} \ . \tag{4}$$

We note that interchanging what we call up and down, and switching the roles of L^i and l^i, is equivalent to complementing the variables x_t^i (i.e., the affine transformation $x_t^i \mapsto 1 - x_t^i$), which maps (3) to (4).

In Section 1, we provide a complete linear-inequality description of the polytope determined by the min-up/min-down restrictions. In Section 2, we provide a very efficient separation procedure for using these inequalities in a branch-and-cut approach to unit-commitment problems. In Section 3, we establish a connection between our polytope and a generalization of the well-known cross polytope (i.e., generalized octahedron).

Notation and terminology: We use $\mathbf{0}$ to denote the 0-vector, $\mathbf{e}$ to denote the all-1 vector, $\mathbf{e}^t$ to denote the t-th standard unit vector, and $\langle \cdot, \cdot \rangle$ to denote the standard dot product in $\mathbb{R}^T$. Transpose signs are ommitted. Finally, if a point satisfies an inequality as an equation, then we say that the point is *tight* for the inequality.

1 Inequality Description

Although the inequalities (3–4) are enough to capture the logical relationships of the 0/1 variables x_t^i that are required by the min-up and min-down restrictions, we can tighten the linear-programming relaxation considerably. Since our inequalities focus on one machine, we suppress the superscript i. We let $P_T(L, \ell)$ be the convex hull (in $\mathbb{R}^T$) of the 0/1 solutions of (3–4) (with i suppressed). Our goal is to find all of the facet-describing inequalities of $P_T(L, \ell)$.

For a nonnegative integer k, consider a nonempty set of $2k + 1$ indices from the discrete interval $[1, T]$:

$$\phi(1) < \psi(1) < \phi(2) < \psi(2) < \cdots < \phi(k) < \psi(k) < \phi(k+1) \ ,$$

such that $\phi(k+1)-\phi(1) \leq L$. We associate with these indices the *alternating up inequality*

$$-\sum_{j=1}^{k+1} x_{\phi(j)} + \sum_{j=1}^{k} x_{\psi(j)} \leq 0 \ . \tag{5}$$

Proposition 1 *The alternating up inequalities are valid for $P_T(L,\ell)$.*

Proof: We simply note that if a pair of variables in the inequality are both equal to 0, then all of the variables with subscripts in between must also be equal to 0. So the 0 values always comprise a single string of variables in the inequality. So, for any feasible solution, the $2k+1$ variables in the inequality must take on values of the form

$$(\overset{\phi(1)}{1}, \overset{\psi(1)}{1}, \ldots, 1, 1, \underbrace{0, 0, \ldots\ldots\ldots, 0, 0}_{\text{a single string of 0's}}, 1, 1, \ldots, \overset{\psi(k)}{1}, \overset{\phi(k+1)}{1}). \tag{6}$$

Since, both strings of 1's (even if one or both are empty), contain at least as many ϕ variables as ψ variables, the total number of ψ variables equal to 1 cannot exceed the number of ϕ variables set to 1. But this is exactly what the inequality (5) says. □

We also have the *alternating down inequality*

$$\sum_{j=1}^{k+1} x_{\phi(j)} - \sum_{j=1}^{k} x_{\psi(j)} \leq 1 \ , \tag{7}$$

where we require $\phi(k+1)-\phi(1) \leq \ell$ (rather than L).

By symmetry, we have the following result.

Proposition 2 *The alternating down inequalities are valid for $P_T(L,\ell)$.*

Note that the simple lower (resp., upper) bound inequalities $-x_j \leq 0$ (resp., $x_j \leq 1$) are alternating up (resp., down) inequalities.

In what follows, it will be convenient to have certain standard points in $\mathbb{R}^T$ at our disposal. For i in $[1,T]$, we define $\mathbf{v}^{(i)} \in \mathbb{R}^T$ by

$$\mathbf{v}_t^{(i)} = \begin{cases} 0, & \text{for } t \text{ in } [1,i]; \\ 1, & \text{for } t \text{ in } (i,T]. \end{cases}$$

Similarly, for i in $[1, T]$, we define $\mathbf{w}^{(i)} \in \mathbb{R}^T$ by

$$\mathbf{w}_t^{(i)} = \begin{cases} 1, & \text{for } t \text{ in } [1, i]; \\ 0, & \text{for } t \text{ in } (i, T]. \end{cases}$$

We note that the $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are always in $P_T(L, \ell)$.

Theorem 3 *The alternating up and down inequalities describe facets of the polytope $P_T(L, \ell)$.*

Proof: We show that the alternating up inequality (5) describes a facet of $P_T(L, \ell)$ by the direct method. The following points T points $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are affinely independent and are tight for (5). These points are of two types. For notational convenience, we let $\psi(0) = 1$ and $\psi(k+1) = T+1$. First, for all h satisfying $1 \leq h \leq k+1$ and all i in $[\phi(h), \psi(h))$, we take $\mathbf{v}^{(i)}$. Secondly, for all h satisfying $0 \leq h \leq k$ and all i in $[\psi(h), \phi(h+1))$, we take $\mathbf{w}^{(i)}$.

As an illustration, these points are shown, for $T = 18$ with $\phi(1) = 3, \psi(1) = 7, \phi(2) = 10, \psi(2) = 13$ and $\phi(3) = 17$, as the rows of the matrix in Figure 1.

It straightforward to check that these T points are: (a) in $P_T(L, \ell)$, (b) tight for (5), and (c) affinely independent:

For (a), we observe that in each point, we switch states at most once, so we cannot violate any min-up or min-down restrictions.

For (b), we observe that for each point, the variable in (5) having minimum (or maximum) index among all variables having value 0 is always a ϕ variable. Therefore, for each point, there are an equal number of ϕ and ψ variables having value 1.

For (c), we consider choosing λ_i and μ_i, such that $\sum_i \lambda_i + \sum_i \mu_i = 0$ and $\sum_i \lambda_i v_t^{(i)} + \sum_i \mu_i w_t^{(i)} = \mathbf{0}$. Considering the variable x_1 (i.e., the first column in Figure 1), we see that $\sum_i \mu_i = 0$, and hence, also $\sum_i \lambda_i = 0$. Then, working through the variables (i.e., columns) in order, we can infer that all $\lambda_i = 0$, and, likewise, all $\mu_i = 0$. Therefore, the points are affinely independent.

We may conclude that the alternating up inequalities describe facets of $P_T(L, \ell)$.

The affine transformation $\mathbf{x} \mapsto \mathbf{e} - \mathbf{x}$ exchanges faces described by alternating up inequalities with faces described by alternating down inequalities (also exchanging L and l). Since the transformation is invertible, it preserves dimension. Therefore, the alternating down inequalities also describe facets of $P_T(L, \ell)$. $\square$

Next, we present our main result.

$$
\begin{array}{r}
\\
\mathbf{v}^{(3)} \\ \mathbf{v}^{(4)} \\ \mathbf{v}^{(5)} \\ \mathbf{v}^{(6)} \\ \mathbf{v}^{(10)} \\ \mathbf{v}^{(11)} \\ \mathbf{v}^{(12)} \\ \mathbf{v}^{(17)} \\ \mathbf{v}^{(18)} \\ \\
\mathbf{w}^{(1)} \\ \mathbf{w}^{(2)} \\ \mathbf{w}^{(7)} \\ \mathbf{w}^{(8)} \\ \mathbf{w}^{(9)} \\ \mathbf{w}^{(13)} \\ \mathbf{w}^{(14)} \\ \mathbf{w}^{(15)} \\ \mathbf{w}^{(16)}
\end{array}
\begin{array}{cccccccccccccccccc}
x_1 & x_2 & x_{\phi(1)} & x_4 & x_5 & x_6 & x_{\psi(1)} & x_8 & x_9 & x_{\phi(2)} & x_{11} & x_{12} & x_{\psi(2)} & x_{14} & x_{15} & x_{16} & x_{\phi(3)} & x_{18} \\
\end{array}
$$

$$
\left(
\begin{array}{cccccccccccccccccc}
0 & 0 & |0 & 1 & 1 & 1| & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & |0 & 0 & 1 & 1| & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & |0 & 0 & 0 & 1| & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & |0 & 0 & 0 & 0| & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & |0 & 1 & 1| & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & |0 & 0 & 1| & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & |0 & 0 & 0| & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & |0 & 1| \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & |0 & 0| \\
 & & & & & & & & & & & & & & & & & \\
|1 & 0| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
|1 & 1| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & |1 & 0 & 0| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & |1 & 1 & 0| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & |1 & 1 & 1| & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & |1 & 0 & 0 & 0| & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & |1 & 1 & 0 & 0| & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & |1 & 1 & 1 & 0| & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & |1 & 1 & 1 & 1| & 0 & 0
\end{array}
\right)
$$

Figure 1: Affinely independent tight points for (5) for $P_T(L, \ell)$ with $T = 18$, where $\phi(1) = 3, \psi(1) = 7, \phi(2) = 10, \psi(2) = 13$ and $\phi(3) = 17$.

Theorem 4 *The alternating up and down inequalities provide a complete linear-inequality description of $P_T(L, \ell)$.*

The proof of Theorem 4 is a direct corollary of the following lemma showing that any point satisfying all of the alternating up and down inequalities can be expressed as a particular convex combination of extreme points of $P_T(L, \ell)$. The following definitions are used in the statement and in the proof of the lemma.

Let $Q_T(L, \ell)$ be the polytope that is the solution set of all of the alternating up and down inequalities. Recall that the integer points in $Q_T(L, \ell)$ are exactly the extreme points of $P_T(L, \ell)$. Let $\mathbf{x}$ be an integer point in $Q_T(L, \ell)$, and let j be in $(1, T]$. The point $\mathbf{x}$ is *put down last at* j if $x_{j-1} = 1$, and $x_k = 0$ for all k in $[j, T]$. Symmetrically, the point $\mathbf{x}$ is *put up last at* j if $x_{j-1} = 0$ and $x_k = 1$ for all k in $[j, T]$. This is indicated in Figures 2 and 3.

$$(\underbrace{\overset{1}{*}, *, \ldots, \overset{j-2}{*}}_{\text{any } 0/1\text{'s}}, \overset{j-1}{1}, \underbrace{\overset{j}{0}, 0, \ldots\ldots\ldots, 0, \overset{T}{0}}_{\text{all } 0\text{'s}}).$$

Figure 2: Put down last at j in $(1, T]$

$$(\underbrace{\overset{1}{*}, *, \ldots, \overset{j-2}{*}}_{\text{any } 0/1\text{'s}}, \overset{j-1}{0}, \underbrace{\overset{j}{1}, 1, \ldots\ldots\ldots, 1, \overset{T}{1}}_{\text{all } 1\text{'s}}).$$

Figure 3: Put up last at j in $(1, T]$

Other than $\mathbf{0}$ and $\mathbf{e}$, each integer point of $Q_T(L, \ell)$, is either put down last or put up last at some j (not both). The points $\mathbf{0}$ and $\mathbf{e}$ are neither put down last nor put up last at any j.

Let $\mathbf{y}$ be an integer point in $Q_{T-1}(L, \ell)$. For $\alpha = 0$ or 1, *extending* $\mathbf{y}$ *with* α is the operation yielding the vector $\mathbf{y}' = (\mathbf{y}, \alpha)$. For $\alpha = 0$, observe that $\mathbf{y}'$ is an integer point in $Q_T(L, \ell)$ if $\mathbf{y}$ is not put up last at j in $[T - L + 1, T - 1]$. For $\alpha = 1$, observe that $\mathbf{y}'$ is an integer point in $Q_T(L, \ell)$ if $\mathbf{y}$ is not put down last at j in $[T - \ell + 1, T - 1]$.

Consider some points $\mathbf{y}^i \in \mathbb{R}^{T-1}$, for i in some finite index set $\mathcal{I}$. Let $\lambda_i \in \mathbb{R}$, satisfy $\lambda_i > 0$ for $i \in \mathcal{I}$. Note that $\sum_{i \in \mathcal{I}} \lambda_i \mathbf{y}^i$ is a convex combination of the $\mathbf{y}^i$ if it happens that $\sum_{i \in \mathcal{I}} \lambda_i = 1$. Let $\lambda'_i \in \mathbb{R}$ satisfy $\lambda_i \geq \lambda'_i \geq 0$, for

$i \in \mathcal{I}$. Let $\alpha = 0$ or 1, and let $\bar{\alpha} := 1 - \alpha$. The *$\lambda'$-extension of the conical combination* $\sum_{i\in\mathcal{I}} \lambda_i \mathbf{y}^i$ *by* α is

$$\sum_{\substack{i\in\mathcal{I}:\\ \lambda'_i>0}} \lambda'_i(\mathbf{y}^i, \alpha) + \sum_{\substack{i\in\mathcal{I}:\\ \lambda_i>\lambda'_i}} (\lambda_i - \lambda'_i)(\mathbf{y}^i, \bar{\alpha})$$

$$= \left(\sum_{i\in\mathcal{I}} \lambda_i \mathbf{y}^i \ , \ \alpha \sum_{\substack{i\in\mathcal{I}:\\ \lambda'_i>0}} \lambda'_i + \bar{\alpha} \sum_{\substack{i\in\mathcal{I}:\\ \lambda_i>\lambda'_i}} (\lambda_i - \lambda'_i) \right) .$$

Observe that this last expression shows that the result of a λ'-extension by α does not depend on the individual values of the λ'_i, but only on their sum. In the remainder of the paper, λ'-extensions are sometimes described by only giving the value of the sum of the λ'_i .

Notice that

$$\sum_{\substack{i\in\mathcal{I}:\\ \lambda'_i>0}} \lambda'_i + \sum_{\substack{i\in\mathcal{I}:\\ \lambda_i>\lambda'_i}} (\lambda_i - \lambda'_i) = \sum_{i\in\mathcal{I}} \lambda_i \ ,$$

so that the extension is a convex combination of the points

$$\left\{ (\mathbf{y}^i, \alpha) \ : \ \lambda'_i > 0 \right\} \cup \left\{ (\mathbf{y}^i, \bar{\alpha}) \ : \ \lambda_i > \lambda'_i \right\}$$

precisely when $\sum_{i\in\mathcal{I}} \lambda_i \mathbf{y}^i$ is a convex combination of the $\mathbf{y}^i$ (i.e., when $\sum_{i\in\mathcal{I}} \lambda_i = 1$). It is easy to see that this extension is also the $(\lambda - \lambda')$-extension of the conical combination $\sum_{i\in\mathcal{I}} \lambda_i \mathbf{y}^i$ by $\bar{\alpha}$.

Lemma 5 *Let $\tilde{\mathbf{x}}$ be a point in $Q_T(L, \ell)$. Then there exist integer points $\mathbf{p}^s$ in $Q_T(L, \ell)$, and $\lambda_s \in \mathbb{R}$, such that*

(i) $\tilde{\mathbf{x}} = \sum_s \lambda_s \ \mathbf{p}^s$, $\sum_s \lambda_s = 1$, *and* $\lambda_s \geq 0 \ \forall \ s$;

(ii) for all j in $[\max\{2, T - L\}, T]$, if $\tilde{x}_j > \tilde{x}_{j-1}$ then $\tilde{x}_j - \tilde{x}_{j-1}$ is exactly the sum of all of the λ_s corresponding to points $\mathbf{p}^s$ put up last at j. Moreover, if $\tilde{x}_j \leq \tilde{x}_{j-1}$, then none of the points $\mathbf{p}^s$ is put up last at j;

(iii) for all j in $[\max\{2, T - \ell\}, T]$, if $\tilde{x}_j < \tilde{x}_{j-1}$ then $\tilde{x}_{j-1} - \tilde{x}_j$ is exactly the sum of all of the λ_s corresponding to points $\mathbf{p}^s$ put down last at j. Moreover, if $\tilde{x}_j \geq \tilde{x}_{j-1}$, then none of the points $\mathbf{p}^s$ is put down last at j.

Proof: Observe that the result for $Q_T(L, \ell)$ and $\tilde{\mathbf{x}}$ is true if and only if it is true for $Q_T(\ell, L)$ and $(\mathbf{e} - \tilde{\mathbf{x}})$, as $(\mathbf{e} - \mathbf{p}^s)$ is an integer point in $Q_T(\ell, L)$ put down (resp., up) last at j if and only if $\mathbf{p}^s$ is put up (resp., down) last at j, and $\mathbf{e} - \tilde{\mathbf{x}} = \sum \lambda_s\ (\mathbf{e} - \mathbf{p}^s)$. Thus we can assume, without loss of generality, that $\tilde{x}_T \geq \tilde{x}_{T-1}$.

We proceed by induction on T. The base case of $T = 1$ is trivial. Now suppose that $T > 1$, and that the result is true for $T - 1$.

Let $\tilde{\mathbf{y}}$ be obtained from $\tilde{\mathbf{x}}$ by deleting its last entry. Note that $\tilde{\mathbf{y}}$ is a point in $Q_{T-1}(L, \ell)$ and thus, by the inductive hypothesis, $\tilde{\mathbf{y}}$ can be written as a convex combination of integer points $\mathbf{a}^s$ in $Q_{T-1}(L, \ell)$,

$$\tilde{\mathbf{y}} = \sum \mu_s\ \mathbf{a}^s, \tag{8}$$

satisfying the corresponding conditions *(i–iii)*.

Note that if $\tilde{x}_T = \tilde{x}_{T-1}$, then for each point $\mathbf{a}^s$, we can define an integer point $\mathbf{p}^s$ in $Q_T(L, \ell)$ by extending $\mathbf{a}^s$ with a 0 if $a^s_{T-1} = 0$ and extending it with a 1 otherwise. Then $\tilde{\mathbf{x}} = \sum \mu_s\ \mathbf{p}^s$. Moreover, conditions *(ii–iii)* for this convex combination are met, as none of the points $\mathbf{p}^s$ is put up last or put down last at $j = T$, and the conditions for smaller indices j are implied by the similar conditions satisfied by the convex combination (8).

Thus, without loss of generality, we have $\tilde{x}_T > \tilde{x}_{T-1}$. We now distinguish two cases:

(a) If $\tilde{y}_{T-\ell} \leq \tilde{y}_{T-\ell+1} \leq \ldots \leq \tilde{y}_{T-1}$, then none of the points $\mathbf{a}^s$ is put down last at j in $[T - \ell + 1, T - 1]$. This means that if $a^s_{T-1} = 0$, then $a^s_j = 0$ for all j in $[T - \ell, T - 1]$. Hence such a point may be used to construct integer points in $Q_T(L, \ell)$ by extending it with a 0 or with a 1. As the sum of the μ_s corresponding to points $\mathbf{a}^s$ with $a^s_{T-1} = 0$ is $1 - \tilde{x}_{T-1} \geq \tilde{x}_T - \tilde{x}_{T-1}$, we can obtain a convex combination of integer points satisfying *(ii–iii)* for $\tilde{\mathbf{x}}$ taking all points $\mathbf{a}^s$ such that $a^s_{T-1} = 1$ and extending them with a 1, taking $\tilde{x}_T - \tilde{x}_{T-1}$ of the points $\mathbf{a}^s$ such that $a^s_{T-1} = 0$ and extending then with a 1, and extending the rest with a 0. Thus, $\tilde{x}$ is a λ'-extension of (8) with $\sum \lambda' = (\tilde{x}_T - \tilde{x}_{T-1}) + \sum\{\mu_s : a^s_{T-1} = 1\}$.

(b) Otherwise, let j_1 be the first index in $[T-\ell, T-2]$ such that $\tilde{x}_{j_1} > \tilde{x}_{j_1+1}$. Now, using the following algorithm, we define indices $j_2 < j_3 < \ldots < j_d = T$ with d odd, such that $\tilde{x}_j$ is nonincreasing on the interval $[j_t, j_{t+1}]$ for t odd, and $\tilde{x}_j$ is nondecreasing on the interval $[j_t, j_{t+1}]$ for t even. (As an illustration, for the case where $x_j \neq x_{j+1}$ for all j in $[T - l, T - 1]$, the indices $j_1, j_3, \ldots$ correspond to local maxima, and $j_2, j_4, \ldots$ to local minima.)

(0) $b := 1$;

(1) While $j_b \neq T$ do

 (1.1) Let $j_{b+1} > j_b$ be the first index such that $\tilde{x}_{j_{b+1}} < \tilde{x}_{1+j_{b+1}}$;

 (1.2) $b := b + 1$;

 (1.3) Let $j_{b+1} > j_b$ be the first index such that $\tilde{x}_{j_{b+1}} > \tilde{x}_{1+j_{b+1}}$;
 if no such index exists, set $j_{b+1} = T$;

 (1.4) $b := b + 1$;

(2) $d := b$;

Note that the index j_{b+1} in step (1.1) always exists as we have $\tilde{x}_{T-1} < \tilde{x}_T$.

Let $1 \leq b < d$. As the convex combination (8) satisfies *(ii–iii)*, the sum of the μ_s associated with points $\mathbf{a}^s$ put down last at j in $(j_b, j_{b+1}]$ is $\tilde{x}_{j_b} - \tilde{x}_{j_{b+1}}$ if b is odd and 0 if b is even. It follows that the sum of the μ_s associated with points $\mathbf{a}^s$ that are put down last at j in $[T - \ell + 1, T - 1]$ is

$$M = \tilde{x}_{j_1} - \tilde{x}_{j_2} + \tilde{x}_{j_3} - \ldots - \tilde{x}_{j_{d-1}}.$$

Observe that $M + \tilde{x}_T$ is the left-hand side of an alternating down inequality of $Q_T(L, \ell)$, implying that $\tilde{x}_T \leq 1 - M$ and thus

$$\tilde{x}_T - \tilde{x}_{T-1} \leq 1 - \tilde{x}_{T-1} - M.$$

Note that $1 - \tilde{x}_{T-1}$ is the sum of the μ_s associated with points $\mathbf{a}^s$ such that $a^s_{T-1} = 0$, and thus $1 - \tilde{x}_{T-1} - M$ is the sum of the μ_s associated with points $\mathbf{a}^s$ such that $a^s_{T-1} = 0$ and $\mathbf{a}^s$ not put down last at j in $[T-\ell+1, T-1]$. Thus, we can obtain a convex combination of integer points satisfying *(ii–iii)*. for $\tilde{x}$ by

- taking all of the points $\mathbf{a}^s$ such that $a^s_{T-1} = 1$ and extending them with a 1;
- taking $\tilde{x}_T - \tilde{x}_{T-1}$ of the points $\mathbf{a}^s$ not put down last at j in $[T-\ell+1, T-1]$ and such that $a^s_{T-1} = 0$ and extending them with a 1;
- extending the rest of the points with a 0.

Indeed, all of the constructed points are in $Q_T(L, \ell)$, none of the constructed points is put down last at $j = T$, and exactly $\tilde{x}_T - \tilde{x}_{T-1}$ of them are put up last at $j = T$. Conditions *(ii–iii)* for smaller indices j are satisfied as the convex combination (8) satisfies them. $\square$

Proof: (Main Theorem) By Lemma 5, every point in $Q_T(L,\ell)$ can be obtained as a convex combination of integer points in $Q_T(L,\ell)$, proving that this polytope is integer. As $P_T(L,\ell) \subseteq Q_T(L,\ell)$ and every integer point in $Q_T(L,\ell)$ is an extreme point of $P_T(L,\ell)$, the two polytopes are identical. $\square$

2 Separation

It is impractical to explicitly generate all of the alternating up and down constraints for an instance of reasonable size. Instead, we wish to do what is common practice in integer programming: Provide an efficient separation algorithm that, for a given $\tilde{\mathbf{x}} \in \mathbb{R}^T$, will determine whether or not $\tilde{\mathbf{x}}$ satisfies all of the alternating up and down inequalities. In the event that the point does not satisfy all of these inequalities, the algorithm will provide a maximally violated inequality. Due to the symmetry between alternating up and down inequalities, we only study the separation of alternating up inequalities. To separate on the alternating down inequalities, substitute $\mathbf{e} - \tilde{\mathbf{x}}$ for $\tilde{\mathbf{x}}$, use ℓ instead of L, and apply the separation algorithm for the alternating up inequalities. If we find a violated alternating up inequality for this transformed situation, we complement variables in the inequality (i.e., substitute $1 - x_t$ for x_t) to get a violated alternating down inequality for the original $\tilde{\mathbf{x}}$.

We first explore properties of maximally violated alternating up inequalities. The following lemma shows, in particular, that we can assume that the indices $\phi(i), \psi(i)$ in the support of a maximally violated inequality are such that $\tilde{x}_j$ is nondecreasing between $\phi(i)$ and $\psi(i+1)$ and nonincreasing between $\psi(i)$ and $\phi(i)$ for $i = 1, \ldots, k$. This observation is the basis of the linear time separation algorithm presented below.

Lemma 6 *If $\tilde{\mathbf{x}}$ satisfying $\mathbf{0} \leq \tilde{\mathbf{x}} \leq \mathbf{e}$ violates an alternating up inequality, then there exists a maximally violated alternating up inequality with support $\phi(1) < \psi(1) < \phi(2) < \psi(2) < \cdots < \phi(k) < \psi(k) < \phi(k+1)$ such that*

(i) $\tilde{x}_{\phi(i)} < \tilde{x}_{\psi(i)} > \tilde{x}_{\phi(i+1)}$ *for* $i = 1, \ldots, k$.

(ii) *For all* $i = 1, \ldots, k$ *and for all* j *in* $(\phi(i), \psi(i))$, *we have* $\tilde{x}_{\phi(i)} < \tilde{x}_j \leq \tilde{x}_{\psi(i)}$ *and* $\tilde{x}_j \leq \tilde{x}_{j+1}$.

(iii) *For all* $i = 1, \ldots, k$ *and for all* j *in* $(\psi(i), \phi(i+1))$, *we have* $\tilde{x}_{\psi(i)} > \tilde{x}_j \geq \tilde{x}_{\phi(i+1)}$ *and* $\tilde{x}_j \geq \tilde{x}_{j+1}$.

(iv) If $\phi(k+1) - \phi(1) < L$*, then either* $\phi(k+1) = T$ *or* $\tilde{x}_{\phi(k+1)} < \tilde{x}_{\phi(k+1)+1}$.

Proof: Consider a maximally violated alternating up inequality with support $\phi(1) < \psi(1) < \phi(2) < \psi(2) < \cdots < \phi(k) < \psi(k) < \phi(k+1)$.

(i): If $\tilde{x}_{\phi(i)} \geq \tilde{x}_{\psi(i)}$ for some i, then removing these two indices from the support of the inequality gives an alternating up inequality whose violation is not smaller than the violation of the original inequality. A similar reasoning shows that we can assume $\tilde{x}_{\psi(i)} > \tilde{x}_{\phi(i+1)}$ for $i = 1, \ldots, k$.

(ii): Suppose that $\tilde{x}_{\phi(i)} \geq \tilde{x}_j$ for some j in $(\phi(i), \psi(i))$. Then replacing $\phi(i)$ by j in the support of the inequality does not decrease its violation. The case $\tilde{x}_j > \tilde{x}_{\psi(i)}$ is similar, but yielding a contradiction as replacing $\psi(i)$ by j would increase the violation. (Note that $\phi(i)$ never decreases, $\psi(i)$ is not changed and that $\phi(i) < \psi(i)$ after replacement).

Suppose that $\tilde{x}_j > \tilde{x}_{j+1}$ for some j in $[\phi(i), \psi(i))$. Then $j \neq \phi(i)$ and $j + 1 \neq \psi(i)$, and adding both j and $j + 1$ in the support of the inequality would increase its violation, a contradiction.

(iii): similar to *(ii)*. (Note that $\psi(i)$ never decreases, $\phi(i + 1)$ is not changed and that $\psi(i) < \phi(i + 1)$ after replacement).

(iv): If the statement does not hold, then replacing $\phi(k+1)$ by $\phi(k+1)+1$ in the support of the inequality does not decrease its violation.

As the modifications of the support of the inequality described in *(i–iv)* either decrease the cardinality of the support or increase the value of some index in the support, only a finite number of such modifications can be performed before an inequality satisfying the statement of the lemma is obtained. □

Observe that if only the value $\phi(1)$ is given for a maximally violated alternating up inequality satisfying the conditions of Lemma 6, then the values of the remaining indices are easy to find: If it exists, $\psi(1)$ will be the smallest index j in $(\phi(1), \phi(1) + L)$ such that $\tilde{x}_j > \tilde{x}_{j+1}$ (due to point *(ii)* of Lemma 6), and then $\phi(2)$ will be the smallest index j in $(\psi(1), \phi(1) + L]$ such that $\tilde{x}_j < \tilde{x}_{j+1}$ or $\phi(2) = \phi(1) + L$ if no such index exists (due to points *(iii-iv)* of Lemma 6). Continuing in this fashion, until no additional ψ index can be found, we can build the support of the inequality. This justifies the algorithm of Figure 4.

The first part of the algorithm computes all of the breakpoints m_r, M_r that can appear in the support of an alternating up inequality satisfying Lemma 6, with m_r corresponding to ϕ indices and M_r to ψ indices. It also compute

```
j := 1;  r := 1;  sum[0] = 0;
While j < T, do
    While j < T and x̃_j ≥ x̃_{j+1}, do j := j + 1;
    m_r := j;  j := j + 1;
    If j ≤ T, then
        While j < T and x̃_j ≤ x̃_{j+1}, do j := j + 1;
        M_r := j;  j := j + 1;
        sum[r] := x̃_{M_r} − x̃_{m_r} + sum[r − 1];
        r := r + 1;
best_lhs := 0;  best_index := 0;
For s = 1 to r − 1, do
    For t = m_s to M_s − 1, do
        t' = min {t + L, T};
        If t' ∈ (m_u, M_u] for some u = s, ..., r − 1, then t' := m_u;
        If t' > t then
            t' ∈ (M_{u−1}, m_u], for some u = s + 1, ..., r − 1;
            lhs := sum[u − 1] − sum[s − 1] − x̃_t + x̃_{m_s} − x̃_{t'};
            if lhs > best_lhs, then
                best_lhs := lhs;  best_index := t;
return(best_index);
```

Figure 4: Alternating up inequality separation

the partial sums

$$sum[k] := \sum_{i=1}^{k}(-\tilde{x}_{m_i} + \tilde{x}_{M_i}) \ .$$

Then it loops on all possible values t that could be used as the value of $\phi(1)$, according to Lemma 6. Once t is known, the last index of a variable in the support of the inequality will either be $t' = \min\ \{t+L, T\}$ if this choice of t' is in $(M_{u-1}, m_u]$ for some u (as t' corresponds to a ϕ index, we must have t' in an interval where $\tilde{x}_j$ is nonincreasing), or if t' is in $(m_u, M_u]$ for some u, then the last index will be m_u. The algorithm returns the value $\phi(1)$ of a most violated alternating up inequality or returns 0 if no such inequality is found.

Figure 5 illustrates the separation algorithm for $L = 12$ and $T = 22$. After computing $m_1 = 3$, $M_1 = 8$, $m_2 = 13$, $M_2 = 18$, $m_3 = 20$, $M_3 = 22$, the inequality generated for $t = 4$ has $t' = 13$ and is $-x_4 + x_8 - x_{13} \leq 0$

with violation 0.25. A most violated up inequality is $-x_3 + x_8 - x_{13} \leq 0$ with violation 0.5.

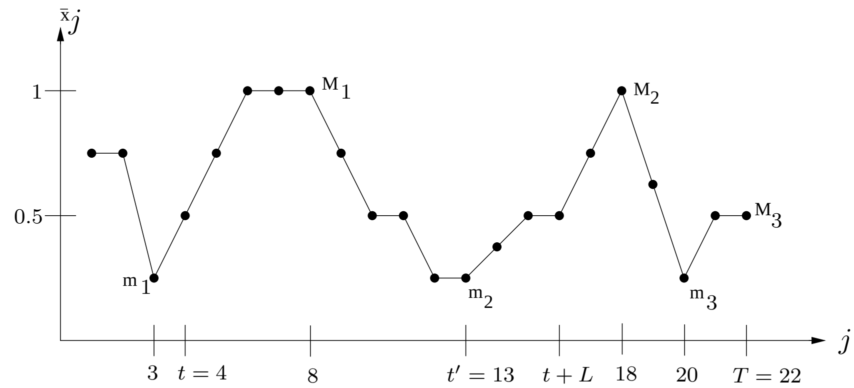

Figure 5: Illustration of the separation algorithm for alternating up inequalities (see Fig. 4) for $L = 12$ and $T = 22$

The running time of this algorithm is $O(T)$, as testing if t' is in $(m_u, M_u]$ for some $u = s, \ldots, r-1$ can be done in constant time once a lookup table $\mathbf{V}$ is constructed (in $O(T)$ time) such that

$$\mathbf{V}[t'] = \begin{cases} u, & \text{if } t' \in (m_u, M_u]; \\ 0, & \text{otherwise.} \end{cases}$$

3 A more symmetric view

Despite the fact that the polytope $P_T(L, \ell)$ has a relatively simple description, it does not appear to belong to a standard class of integral polytopes. In this section, we describe a certain invertible affine transformation ρ of the polytope $P_T(L, \ell)$. This affine transformation affords us with a somewhat more symmetric view. The motivation for the transformation is that it takes $P_T(T-1, T-1)$ into the well-known *cross polytope* (or *hyperoctahedron*, or *unit ball of the ℓ_1 norm)*

$$
\begin{aligned}
C_T^* &:= \operatorname{conv}\left\{\pm \mathbf{e}^t \ : \ t \in [1,T]\right\} \\
&= \left\{ y \in \mathbb{R}^T \ : \ \sum_{i=1}^{T} |y_t| \le 1 \right\} \\
&= \left\{ y \in \mathbb{R}^T \ : \ \sum_{i=1}^{T} \epsilon_t y_t \le 1, \ \forall \ \epsilon \in \{+,-\}^T \right\}
\end{aligned}
$$

(see, for example (Ziegler, 1995, p. 8), or (Ball, 1997, p. 3)).

The transformation ρ from $x \in P_T(L,\ell)$ to $y \in \mathbb{R}^T$ is defined as follows:

$$
y_t := \begin{cases} x_1 - (1 - x_T), & \text{for } t = 1; \\ x_t - x_{t-1}, & \text{for } t \text{ in } [2,T]. \end{cases}
$$

For periods $t > 1$ we have the following interpretation: $y_t = 1$ indicates that the state has changed from down in period $t-1$ to up in period t; similarly, $y_t = -1$ indicates that the state has changed from up in period $t-1$ to down in period t; finally, $y_t = 0$ indicates that the state in period t is the same as in period $t-1$.

Period 1 is a bit different. The variable y_1 depends on x_1 and x_T. If we artificially define $x_0 = 1 - x_T$ (that is the 0/1-state in "period 0" is the opposite of the state in period T), then y_1 behaves just like the other y_t (i.e., $y_1 = x_1 - x_0$).

The transformation ρ takes $\mathbf{w}^{(t)}$ into $\mathbf{e}^t$ and $\mathbf{v}^{(t)}$ into $-\mathbf{e}^t$, for t in $[1,T]$. In this way, we see that $\rho(P_T(T-1,T-1)) = C_T^*$.

We have ρ^{-1} described by:

$$
x_t := \frac{1}{2}\left(1 + \sum_{j \le t} y_j - \sum_{j > t} y_j\right), \text{ for } t \text{ in } [1,T].
$$

Looking at an example will either create or dispel some confusion. We will take a chance. The simple bound inequalities $0 \le x_t \le 1$ for $P_T(L,\ell)$ translate, using ρ^{-1}, to the inequalities

$$
-1 \le \sum_{j \le t} y_j - \sum_{j > t} y_j \le 1 \ . \tag{9}
$$

In particular, for the hypercube $C_T := \{x \in \mathbb{R}^T \ : \ \mathbf{0} \le x \le \mathbf{e}\} = P_T(1,1)$, we have $\rho(C_T)$ completely described as the solution set of the inequalities (9), for

t in $[1, T]$. In this way, we can view $\rho(P_T(1,1))$ as a *zonotope* (i.e, an affine transformation of a hypercube; see (Ziegler, 1995, p. 198)).

For r, s satisfying $1 \leq r \leq s \leq T$, we define $\mathbf{u}^{(r,s)} \in \mathbb{R}^T$ by

$$\mathbf{u}_t^{(r,s)} = \begin{cases} 1, & \text{for } t \text{ in } [r, s]; \\ 0, & \text{otherwise.} \end{cases}$$

We note that the $\mathbf{u}^{(r,s)}$ are in $P_T(L, \ell)$ whenever $s - r + 1 \geq L$.

Now, consider the point $\mathbf{u}^{(r,s)} \in P_T(L, \ell)$, for some r and s satisfying $1 < r < t < s < T$, and $s - r \geq L$. Moving too quickly, it appears that $\rho(\mathbf{u}^{(r,s)}) = \mathbf{e}^r - \mathbf{e}^s$, which violates (9). This is all cleared up when we realize that $\rho(\mathbf{u}^{(r,s)}) = -\mathbf{e}^1 + \mathbf{e}^r - \mathbf{e}^s$, which satisfies (9).

Using the description of ρ^{-1}, we can transform the alternating up and down inequalities to inequalities that describe $\rho(P_T(L, \ell))$. Of course, we should see the 2^T facet-describing inequalities $\sum_{t=1}^{T} \epsilon_t y_t \leq 1$ of the cross polytope C_T^*, when $L = l = T - 1$. Here is a pretty clean description of the inequalities in general: Each inequality has the form $\sum_{t=1}^{T} \epsilon_t y_t \leq 1$, where $\epsilon_t \in \{+, -\}$, for t in $[1, T]$. Now, these 2^T inequalities are not all valid in general. Consider the sign sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_T$. In the sign sequence, there are the maximal runs of the same sign. Let a be the last index of the first run. There is a last run that has the same sign as the first run; let b be the last index of this run (we will have $b = a$ if there is no other run having the same sign as the first run). For validity we require (i) $b - a \leq L$ if these runs are of $-$'s, and (ii) $b - a \leq l$ if these runs are of $+$'s.

In some ways, $\rho(P_T(L, \ell))$ is more symmetric than $P_T(L, \ell)$. In particular, we have (i) $\rho(P_T(L, \ell)) = -\rho(P_T(\ell, L))$, and the right-hand sides of the inequalities are always 1. Perhaps this symmetry can be exploited to simplify the proof of a complete characterization by linear inequalities.

We make some further observations concerning (T-dimensional) volumes. The linear part of ρ has determinant 2. Therefore, the zonotope $\rho(P_T(1,1) = C_T)$ has volume 2. Also, since the cross polytope C_T^* is known to have volume $2^T/T!$ (see, for example, (Ball, 1997, p. 3)), then $P_T(T-1, T-1) = \rho^{-1}(C_T^*)$ has volume $2^{T-1}/T!$. These observations may serve as a starting point for determining the volume and perhaps even the Ehrhart polynomial for each $P_T(L, \ell)$. It would be interesting to compare, in the manner of (Lee and Morris, 1994), the volume of $P_T(L, \ell)$ with the volume of the relaxation (3–4) employed by (Takriti et al., 2000).

Acknowledgments

The authors thank Samer Takriti for telling them about (Takriti et al., 2000). The work of Jon Lee was partially carried out at and supported by the Chinese University of Hong Kong. The work of Janny Leung was partially carried out at and supported by the IBM T.J. Watson Research Center. The work of François Margot was carried out at and partially supported by the IBM T.J. Watson Research Center.

References

Ball, K. (1997). An elementary introduction to modern convex geometry. In *Flavors of geometry* (pp. 1–58). Cambridge: Cambridge University Press.

Birge, J., Takriti, S. and Long, E. (1995). Intelligent unified control of unit commitment and generation allocation: Final report EPRI Grant RP8030-13. *Unpublished manuscript.*

Lee, J. and Morris, W. D., Jr. (1994). Geometric comparison of combinatorial polytopes. *Discrete Applied Mathematics*, *55*(2), 163–182.

Nemhauser, G. L. and Wolsey, L. A. (1999). *Integer and combinatorial optimization.* New York: John Wiley & Sons Inc. (Reprint of the 1988 original, A Wiley-Interscience Publication)

Rothkopf, M. H., Hobbs, B. F., O'Neill, R. P. and Chao, H.-P. (2001). *The Next Generation of Electric Power Unit Commitment Models.* International Series in Operations Research and Management Science, Volume 36. Kluwer. (Conference web page: `dimacs.rutgers.edu/Workshops/NextGeneration/program.html`)

Takriti, S., Krasenbrink, B. and Wu, L. S.-Y. (2000). Incorporating fuel constraints and electricity spot prices into the stochastic unit commitment problem. *Operations Research*, *48*(2), 268–280.

Ziegler, G. M. (1995). *Lectures on polytopes* (Vol. 152). New York: Springer-Verlag. (Revised edition, 1998)