Weak k-majorization and polyhedra

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Abstract

For integers k and n with $k \leq n$ a vector $x \in \mathbf{R}^n$ is said to be weakly k-majorized by a vector $q \in \mathbf{R}^k$ if the sum of the r largest components of x does not exceed the sum of the r largest components of q, for $r = 1, \ldots, k$. For a given q the set of vectors weakly k-majorized by q defines a polyhedron $P(q;k)$. We determine the vertices of both $P(q;k)$ and its integer hull $Q(q;k)$. Furthermore a complete and nonredundant linear description of $Q(q;k)$ is given.
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1 Introduction

In many branches of mathematics and statistics *majorization* plays a role in establishing inequalities between e.g., eigenvalues, singular values etc. The basic notion of majorization reflects to what extent components of vectors are "spread out". For $p, q \in \mathbf{R}^n$ one says that p is *weakly sub-majorized* by q if $\sum_{j=1}^r p_{[j]} \leq \sum_{j=1}^r q_{[j]}$ for $r = 1, \ldots, n$. Here $p_{[j]}$ denotes the j'th largest component of p. If also $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$ holds, p is *majorized* by q and we write $p \prec q$. Several equivalent conditions for (weak sub-) majorization are known (see [7]). For instance, using the Birkhoff-von Neumann theorem (see e.g., [7] or [10]), one can show that $p \prec q$ iff there is a doubly stochastic matrix $M \in \mathbf{R}^{n,n}$ (i.e., M has nonnegative elements and all row and column sums are 1) with $p = Mq$. As a consequence, $p \prec q$ if and only if p lies in the convex hull of the set of vectors obtained by permuting the components of q. A similar characterization holds for weak submajorization. An extensive treatment of the theory of majorization as well as its applications in e.g. matrix theory, numerical analysis and statistics is

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given in the book by Marshall and Olkin [7]. For generalizations of majorization within a measure theoretical framework as well as statistical interpretations, see the extensive treatment in [11]. In [1] approximate majorization is studied.

A function $\phi : \mathbf{R}^n \to \mathbf{R}$ that preserves the ordering given by majorization is called *Schur-convex*, thus $\phi(x) \leq \phi(q)$ whenever $x \prec q$. Therefore q maximizes $\phi(x)$ over the set $x \prec q$. A general and important technique for finding inequalities in various fields is to discover some underlying majorization combined with a suitable Schur-convex function. A simple inequality obtained in this way, which is useful in this work, is the *rearrangement inequality* due to Hardy, Littlewood and Polya, see [6], [7]. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then we have:

$$\sum_{i=1}^{n} a_{[i]} b_{[n-i+1]} \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_{[i]} b_{[i]}. \tag{1}$$

In this paper we study *weak k-majorization* in which we relax the partial sum constraints of weak sub-majorization for $r > k$. The main goal is to investigate certain polyhedra associated with this notion. Several properties of these polyhedra are established. We should point out that the main results of this work were presented in [2], but using a different approach. This paper is organized as follows. In Section 2 we introduce weak k-majorization and describe some of its basic properties. The vertices of different majorization polyhedra are studied in Section 3 while in the next section we study the convex hull of all the integral vectors satisfying a weak k-majorization constraint.

Notation. $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{Q}$ denote the set of real, integral and rational numbers, respectively. For $1 \leq a \leq b \leq n$ and v in $\mathbf{R}^n$, we define $v_{a:b} := \sum_{j=a}^{b} v_j$ and $\bar{v}_{a:b} := v_{a:b}/(b-a+1)$. Note that $\bar{v}_{a:b}$ is simply the average of the components $v_a, \ldots, v_b$. For each positive integer t we let $\mathbf{N}_t := \{1, \ldots, t\}$, and for $x \in \mathbf{R}^n$, $x_{[j]}$ is the j'th largest component of x. When $S \subseteq \mathbf{N}_n$ we let $|S|$ denote the cardinality of S, and if $x \in \mathbf{R}^n$ we define $x(S) := \sum_{j \in S} x_j$. For concepts and results concerning polyhedra and linear inequalities, see [8] or [10]. When π is a permutation on $\mathbf{N}_n$ (i.e., a bijection) and $a \in \mathbf{R}^n$ we call the vector $(a_{\pi(1)}, \ldots, a_{\pi(n)})$ a *permutation* of a. A set $A \subseteq \mathbf{R}^n$ is *symmetric* if it contains each permutation of its vectors. We let $e_i \in \mathbf{R}^n$ be the i'th unit (coordinate) vector in $\mathbf{R}^n$, i.e., the i'th component of e_i is 1 and all other components are 0. We also let $[a, b] := \{x \in \mathbf{R} \mid a \leq x \leq b\}$.

2 Weak k-majorization and optimization

We introduce and study basic properties of weak k-majorization. Associated optimization problems and polyhedra are also introduced.

Let k and n be two given integers such that $k \leq n$, and let the *majorant* $q \in \mathbf{R}^k$ be a given vector satisfying $q_1 \geq \ldots \geq q_k$. We say that $x \in \mathbf{R}^n$ is *weakly*

k-*majorized* by q and write $x \prec_k q$ if the following conditions hold:

$$\sum_{j=1}^r x_{[j]} \leq \sum_{j=1}^r q_{[j]} \quad \text{for all } r \in \mathbf{N}_k. \tag{2}$$

Note that $x \prec_k q$ iff some permutation of x is weakly k-majorized by some permutation of q. For $k = n$ the notion $\prec_k$ coincides with weak sub-majorization. Furthermore, weak k-majorization corresponds to weak sub-majorization applied to the k largest components of the vectors. The last observation means that equivalent conditions for weak k-majorization may be adopted from that of weak sub-majorization and expressed in terms of the subvectors consisting of the k largest components. One of the results of the present work is to find another characterization of k-majorization expressed in terms of the full vector x.

It is of interest to study optimization problems in connection with weak k-majorization. Let $c \in \mathbf{R}^n$ be a nonnegative objective function vector and consider the problem

$$\max \{c^T x \mid x \prec_k q\}. \tag{3}$$

Here we may interpret c_j as the "expected value" or profit associated with a project $j \leq n$. When the variable x_j represents the investment in project j, the problem (3) is to maximize the total profit of the investments under the requirement that investments are "suitably spread out" (which reduces the overall risk).

Let $P(q;k) := \{x \in \mathbf{R}^n \mid x \prec_k q\}$ be the feasible set of (3). Note that $x \prec_k q$ if and only if $x(S) \leq q(\mathbf{N}_r)$ for each subset S of $\mathbf{N}_n$ with $|S| = r \leq k$ because the maximum value of $x(S)$ taken over all such subsets is $\sum_{j=1}^r x_{[j]}$. Thus the set $P(q;k)$ is a polyhedron,

$$P(q;k) = \{x \in \mathbf{R}^n \mid x(S) \leq q(\mathbf{N}_r) \text{ for all } S \subseteq \mathbf{N}_n \text{ with } r = |S| \leq k\}. \tag{4}$$

The polyhedron $P(q;k)$, called a *majorization polyhedron*, is unbounded, its characteristic cone is $\mathbf{R}^n_-$ and it is pointed, i.e., its minimal faces are vertices. Furthermore, $P(q;k)$ is symmetric. Note also that a nonincreasing vector v in $\mathbf{R}^n$ is in $P(q;k)$ if and only if $v_{1:j} \leq q_{1:j}$ for all $1 \leq j \leq k$.

Each n-majorization polyhedron may be viewed as a *polymatroid* (see e.g. [4], [5]) associated with the set function $f(S) = \sum_{j=1}^r q_j$ for each $S \subseteq \mathbf{N}_n$ where $r := |S|$. (Trivially, this function is monotone and submodular.) Thus (see [3]) (3) may be solved by the greedy algorithm and the optimal solution (when $c_1 \geq \ldots \geq c_n \geq 0$) is $x = q$. This result also follows easily from (1). Some further properties in the case $k = n$ are discussed in [2]. For $k < n$, however, $P(q;k)$ may not be a polymatroid and therefore the greedy solution which is $x_j = q_j$ for $j \leq k$ and $x_j = q_k$ for $j > k$ may not be optimal for (3). For instance, with $n = 3$, $k = 2$, $q = (2, 1)$ and $c = (1, 1, 1)$ the greedy algorithm produces the nonoptimal solution $(2, 1, 1)$ while the optimal solution is $(3/2, 3/2, 3/2)$. Therefore it is clear that there are other vertices of $P(q;k)$

than the permutations of q. In the next section all the remaining vertices are described.

We also consider the integer linear programming problem corresponding to (3), or equivalently, the problem of maximizing $c^T x$ over the integer hull of $P(q;k)$ which is the following polyhedron

$$Q(q;k) = \text{conv}(\{x \in \mathbf{R}^n \mid x \prec_k q, x \text{ is integral}\}). \tag{5}$$

These optimization problems are motivated by applications where some sort of diversification is required. For instance, assume that (indivisible) jobs are to be allocated to n different computers in some network. Let x_j be the number of jobs to be performed on computer j, and assume that c_j is the cost per job on computer j. A weak k-majorization constraint on the vector $x = (x_1, \ldots, x_n)$ assures that jobs are suitably spread out among the computers for e.g. security reasons. The mentioned integer linear programming problem reflects the problem of minimum cost job allocation subject to the security constraint. In more realistic models the allocation would also be subject to other kinds of constraints.

3 Vertices of majorization polyhedra

We study the inner description of the polyhedra $P(q;k)$ and $Q(q;k)$. In the sequel, we assume that $k < n$.

Let $\alpha \in [q_k, \bar{q}_{1:k}]$ and define the numbers $s(\alpha) = \max\{0 \leq s < k \mid \bar{q}_{s+1:k} \geq \alpha\}$ and $\Delta(\alpha) = q_{s(\alpha)+1:k} - (k - s(\alpha) - 1)\alpha$. We also define the vector $x(\alpha) \in \mathbf{R}^n$ by

$$x(\alpha) = (q_1, \ldots, q_{s(\alpha)}, \Delta(\alpha), \alpha, \ldots, \alpha). \tag{6}$$

We will show that each extreme point of $P(q;k)$ or $Q(q;k)$ is a permutation of $x(\alpha)$ for particular values of α. Some useful properties of $x(\alpha)$ are contained in the following lemma. They imply in particular that $x(\alpha)$ is in $P(q;k)$.

Lemma 3.1 *For each $\alpha \in [q_k, \bar{q}_{1:k}]$ we have that*

(i) $x(\alpha)_{1:k} = q_{1:k}$,

(ii) $\Delta(\alpha) \geq \alpha$,

(iii) $q_{s(\alpha)+1} \geq \Delta(\alpha)$ *with equality if and only if* $s(\alpha) = k - 1$, *and*

(iv) $x(\alpha)$ *is nonincreasing and* $x(\alpha) \prec_k q$.

Proof. Property (i) holds since the definition of $\Delta(\alpha)$ yields directly that $x(\alpha)_{s(\alpha)+1:k} = q_{s(\alpha)+1:k}$. Since $\bar{q}_{s(\alpha)+1:k} \geq \alpha$ implies $q_{s(\alpha)+1:k} \geq (k - s(\alpha))\alpha$, (ii) is true. Note that the definition of $s(\alpha)$ implies that

$$q_{t:k} < (k - t + 1)\alpha \quad \text{for all} \quad s(\alpha) + 2 \leq t \leq k. \tag{7}$$

If $s(\alpha) = k-1$ then $\Delta(\alpha) = q_k$ and both (iii) and (iv) hold. Otherwise, $\Delta(\alpha) = q_{s(\alpha)+1} + q_{s(\alpha)+2:k} - (k - s(\alpha) - 1)\alpha$ and together with (7) for $t = s(\alpha)+2$ this yields that $\Delta(\alpha) < q_{s(\alpha)+1}$, proving (iii). With (ii) and (iii), we have that $x(\alpha)$ is nonincreasing. Finally, Property (i) together with (7) imply that $x(\alpha)_{1:t-1} < q_{1:t-1}$ for all $s(\alpha) + 2 \leq t \leq k$ and (iv) follows.

□

The following lemma leads to a description of the vertices of majorization polyhedra.

Lemma 3.2 *Let $c \geq 0$ be a nonincreasing vector in $\mathbf{R}^n$, let $\alpha \in [q_k, \bar{q}_{1:k}]$ and consider the problem*

$$\max \{c^T x \mid x \prec_k q, x_{[k]} = \alpha\}. \tag{8}$$

Then $x(\alpha)$ is an optimal solution of (8). Furthermore, if α_1 and α_2 satisfy $\bar{q}_{j:k} \geq \alpha_1 \geq \alpha \geq \alpha_2 \geq \bar{q}_{j+1:k}$ for some $1 \leq j < k$, then $x(\alpha)$ is a convex combination of $x(\alpha_1)$ and $x(\alpha_2)$.

Proof. Let x be a nonincreasing optimal solution to (8), let $t := \max\{0 \leq i \leq k-1 \mid x_i > \alpha\}$ and let $t' := \max\{0 \leq i \leq k-1 \mid x(\alpha)_i > \alpha\}$. One may suppose w.l.o.g. that x is chosen among the optimal solutions so that t is minimum.

We first show that $t \leq t'$. For suppose $t' < t$ and let $d(t) := q_{t:k} - (k-t)\alpha$. Note that $d(t) \leq q_t$ since $(k-t)\alpha = x(\alpha)_{t+1:k} \geq q_{t+1:k}$ the last inequality being implied by the feasibility of $x(\alpha)$ and Lemma 3.1 (i). Let $x' := (x_1, \ldots, x_t)$, $c' := (c_1, \ldots, c_t)$ and $q' := (q_1, \ldots, q_{t-1}, d(t))$. Note that q' is nonincreasing, that x' is a feasible solution to $\max \{(c')^T y \mid y \prec q'\}$ and that q' is an optimal solution since the greedy algorithm solves this problem to optimality (see Section 2). If $d(t) \leq \alpha$ then there exists $1 > \lambda \geq 0$ such that for $y = \lambda x' + (1-\lambda)q'$ we have $y_t = \alpha$. Then $y^* = (y_1, \ldots, y_t, \alpha, \ldots, \alpha) \in \mathbf{R}^n$ is a feasible solution to (8). Moreover, as $c'^T q' \geq c'^T x'$, we have $c^T y^* \geq c^T x$ and thus y^* is an optimal solution to (8). But this contradicts the choice of x, since $t' < t$ and y^* is a nonincreasing optimal solution to (8) with $y^*_{t'} = \ldots = y^*_k = \alpha$. Suppose now that $d(t) > \alpha$, i.e., $q_{t:k} > (k-t+1)\alpha$ implying $s(\alpha) \geq t-1$ and $q_t > \alpha$. The definition of $x(\alpha)$ and Lemma 3.1 show that either (a) $s(\alpha) = t'-1$ or (b) $s(\alpha) = t'$ and $\Delta(\alpha) = \alpha$ or (c) $s(\alpha) = k-1$ and $q_{t'+1} = \ldots = q_k = \alpha$. Note that (a) would imply $s(\alpha) = t'-1 < t-1 \leq s(\alpha)$, a contradiction; a similar argument shows that (b) would imply $s(\alpha) = t' = t-1$ and thus $\Delta(\alpha) = d(t)$, a contradiction since $\Delta(\alpha) = \alpha$ and $d(t) > \alpha$. Finally, (c) is impossible since we have $q_t > \alpha$ as noted above.

Hence we have $t \leq t'$. Let $\bar{x} := (x_1, \ldots, x_{t'})$, $\bar{c} := (c_1, \ldots, c_{t'})$ and $\bar{q} := (q_1, \ldots, q_{t'-1}, x(\alpha)_{t'})$. Observe that $\bar{x}$ is a feasible solution to $\max \{\bar{c}^T y \mid y \prec \bar{q}\}$ and that $\bar{q}$ is an optimal solution. Since $\bar{q}$ is the vector containing the first t' components of $x(\alpha)$, we have $c^T x(\alpha) \geq c^T x$, proving that $x(\alpha)$ is an optimal solution to (8).

The last statement of the lemma is trivially true if $\alpha_1 = \alpha$ or $\alpha_2 = \alpha$. Otherwise, we have $\bar{q}_{j:k} > \alpha > \bar{q}_{j+1:k}$ and thus $s(\alpha) = s(\alpha_1) = j-1$. We distinguish three cases: (a) if $\alpha_2 > \bar{q}_{j+1:k}$ then $s(\alpha_2) = j-1$; (b) if $\alpha_2 = \bar{q}_{j+1:k}$ then either (b.1) $q_{j+1} > \alpha_2$, implying $s(\alpha_2) = j$ and $\Delta(\alpha_2) = \alpha_2$ or (b.2) $q_{j+1} = \alpha_2$, implying $q_{j+1} = \ldots = q_k = \alpha_2$ and $s(\alpha_2) = k-1$. Note that in all three cases, we have $x(\alpha_2)_{j+1} = \ldots = x(\alpha_2)_k = \alpha_2$. It follows that the three vectors $x(\alpha)$, $x(\alpha_1)$ and $x(\alpha_2)$ have the same first $(j-1)$ components and each of them is constant on the components with index $\geq j+1$. Then $x(\alpha) = \lambda x(\alpha_1) + (1-\lambda)x(\alpha_2)$ for $\lambda = (\alpha - \alpha_2)/(\alpha_1 - \alpha_2)$ since it is immediate that this is verified for all components except (possibly) for component j. The result for component j then follows from Lemma 3.1 (i).

□

Lemma 3.3 *Let v be a nonincreasing vector in $\mathbf{R}^u$ and $1 \leq p \leq u$. Then there exists $1 \leq a \leq b \leq p$ such that*

$$v_{1:u}/p > v_{2:u}/(p-1) > \ldots > v_{a:u}/(p-a+1) = \ldots = v_{b:u}/(p-b+1) < \ldots < v_{p:u}.$$

Moreover, we have $b = p$ in these relations whenever $p = u$.

Proof. Let $d_j := v_{j:u}/(p-j+1)$ for all $1 \leq j \leq p$. Then, for all $1 \leq j \leq p-1$, we have $d_j = (1/(p-j+1))v_j + ((p-j)/(p-j+1))d_{j+1}$, i.e., d_j is a convex combination of v_j and d_{j+1}. Moreover, if $v_j > d_{j+1}$, then $v_j > d_j > d_{j+1}$ and the fact that v is nonincreasing then implies that $d_i > d_{i+1}$ for all $1 \leq i \leq j$. As $v_j < d_j$ implies $d_j < d_{j+1}$ and $v_j = d_{j+1}$ implies $d_j = d_{j+1}$, the result follows.

□

For $s = 0, \ldots, k-1$ we let $w^s := x(\bar{q}_{s+1:k}) = (q_1, \ldots, q_s, \bar{q}_{s+1:k}, \ldots, \bar{q}_{s+1:k}) \in \mathbf{R}^n$ and call these vectors *q-averages*. For notational convenience, we also define $w^k := w^{k-1}$. We denote by $W(q;k)$ the set $\{w^s \mid 0 \leq s \leq k\}$. Lemma 3.3 with $u := k$ and $p := k$ shows that there exists $1 \leq s^* \leq k$ such that $\bar{q}_{1:k} > \ldots > \bar{q}_{s^*:k} = \ldots = \bar{q}_{k:k}$ implying that $w^s \neq w^t$ for $0 \leq s < t \leq s^* - 1$. Moreover, as $\bar{q}_{s^*:k} = \ldots = \bar{q}_{k:k}$, we have $q_{s^*} = \ldots = q_k$ and thus $w^{s^*-1} = \ldots = w^{k-1}$. It follows that $W(q;k)$ contains exactly s^* distinct elements, namely $w^0, \ldots, w^{s^*-1}$.

Lemma 3.4 *Let $c \geq 0$ be a nonincreasing vector in $\mathbf{R}^n$. Then there exists $0 \leq a \leq b \leq s^* - 1$ such that*

$$c^T w^0 < \ldots < c^T w^{a-1} < c^T w^a = \ldots = c^T w^b > c^T w^{b+1} > \ldots > c^T w^{s^*-1} \quad (9)$$

and

$$c_{1:n}/k > \ldots > c_{a+1:n}/(k-a) = \ldots = c_{b+1:n}/(k-b) < \ldots < c_{s^*:n}/(k-s^*+1).$$

Proof. Using Lemma 3.3 with $u := n$ and $p := k$ proves the existence of $1 \leq a \leq b \leq s^* - 1$ such that the second inequality of the above statement holds. For $0 \leq j \leq s^* - 2$, $c^T w^j \geq c^T w^{j+1}$ if and only if $c^T r^j \geq 0$ where $r^j := w^j - w^{j+1}$. From the definition of q-averages we see that $r_i^j = 0$ for all $i \leq j$, $r^j_{j+1} = \bar{q}_{j+1:k} - q_{j+1} = (q_{j+1:k} - (k-j) \cdot q_{j+1})/(k-j)$ and finally $r^j_i = \bar{q}_{j+1:k} - \bar{q}_{j+2:k} = ((k-j) \cdot q_{j+1} - q_{j+1:k})/((k-j) \cdot (k-j-1))$ for all $i \geq j+2$.

It follows that $0 \geq r^j_{j+1} = -(k-j-1) \cdot r^j_i$ for all $j+2 \leq i \leq n$. As $r^j_{j+2} > 0$, we have $c^T(w^j - w^{j+1}) \geq 0$ if and only if $-(k-j-1) \cdot c_{j+1} + c_{j+2:n} \geq 0$, i.e., if and only if $c_{j+1} \leq c_{j+2:n}/(k-j-1)$. As shown in the proof of Lemma 3.3, this is equivalent to say that $c_{j+1:n}/(k-j) \leq c_{j+2:n}/(k-j-1)$ and thus the first inequality of the statement follows from the second inequality.

□

The first main result is given next.

Theorem 3.5 *The vertex set of $P(q;k)$ is the set of vectors that can be obtained as a permutation of one of the vectors $w^0, \ldots, w^{s^*-1}$.*

Proof. To prove that each vertex has the desired form, let c be an objective function such that the LP problem (3) has a unique optimal solution $\bar{x}$. Then $c > 0$ and one can suppose w.l.o.g. that $\bar{x}$ is nonincreasing as $P(q;k)$ is symmetric. The rearrangement inequality (1) then shows that c is nonincreasing. We first show that some w^s is optimal for this LP. Since $c > 0$ we have $\bar{x}_k \geq q_k$. Observe that any optimal solution of (8) with $\alpha = \bar{x}_k$ must also be optimal in (3). Thus it follows from Lemma 3.2 that $x(\alpha)$ is optimal in (3). Furthermore, from the second part of Lemma 3.2 we see that we may assume that $\alpha = \bar{q}_{j:k}$ for some j, as desired.

We finally prove that each w^s for $0 \leq s \leq s^* - 1$ is indeed an extreme point of $P(q;k)$. Let $1 > \epsilon > 0$ and consider the cost function c given by $c_i = n + \epsilon^i$ for $1 \leq i \leq s$ and $c_i = 1 + \epsilon^i$ for $s+1 \leq i \leq n$. Lemma 3.4 shows that, for $\epsilon > 0$ small enough, the unique optimum solution to max $\{c^T w^t \mid 0 \leq t \leq s^* - 1\}$ is the q-average w^s. By (1) and the fact that c is stricly decreasing, $c^T w^s > c^T w'$, if $w' \neq w^s$ and w' is obtained by permutation of a vector in $W(q;k)$. Therefore w^s is a vertex of $P(q;k)$.

□

It follows that the linear programming problem (3) may be solved easily by sorting the components of the objective function c in nonincreasing order and comparing the $s^* - 1$ different q-averages. In Fig.**??** we illustrate the intersection between $P(q;k)$ and the nonnegative orthant for $k = 2$, $n = 3$ and $q = (2,1)$. The different permuted q-averages are shown. As another consequence of Theorem 3.5 we obtain an inner description of $P(q;k)$ as well as an equivalent condition for weak k-majorization.

Corollary 3.6 *$P(q;k) = conv(W(q;k)) + \mathbf{R}^n_-$, i.e., $x \prec_k q$ if and only if $x \leq z$ for some $z \in \mathbf{R}^n$ which is a convex combination of permuted q-averages.*

We now turn to a discussion of the polyhedron $Q(q;k)$. Observe that $Q(q;k)$ is unchanged if we perform integer round-down on each component of the majorant q. Thus we may assume that q is integral. Let $m^* = \lfloor \bar{q}_{1:k} \rfloor$ and $m_* = q_k$.

We say that $m \in \{m_*, \dots, m^*\}$ is *q-extreme* if m is obtained by integer rounding (up or down) of some tail average $\bar{q}_{s:k}$ of q. When m is q-extreme we call $x(m)$ a *rounded q-average*. By using similar arguments as in the proof of the first part of Theorem 3.5, we get the following result on the vertices of $Q(q;k)$.

Proposition 3.7 *Each vertex of $Q(q;k)$ may be obtained as a permutation of some rounded q-average.*

Note that the converse of this result is proved in the next section in Theorem 4.11, but in the meantime the above result is sufficient for our purposes. The complete characterization of the vertices of $Q(q;k)$ given in that theorem yields that solving LP problems over $Q(q;k)$ (or, equivalently, integer LP's over $P(q;k)$) may be done by sorting the components of the objective function in nonincreasing order and direct comparison of the rounded q-averages.

Examples. Let $k = 3$, $n = 5$ and $q = (7, 2, 1)$. Then the rounded q-averages are $q^1 = (7, 2, 1, 1, 1)$, $q^2 = (6, 2, 2, 2, 2)$ and $q^3 = (4, 3, 3, 3, 3)$, and the vertices of $Q(q;k)$ are all permutations of these points. As another example let $k = 4$, $n = 6$ and $q = (19, 12, 5, 3)$. Then the tail averages of q are $3, 4, 20/3$ and $39/4$ and the q-extreme integers are 3, 4, 6, 7 and 9. The q-averages are $q^3 = (19, 12, 5, 3, 3, 3)$, $q^4 = (19, 12, 4, 4, 4, 4)$, $q^6 = (19, 8, 6, 6, 6, 6)$, $q^7 = (18, 7, 7, 7, 7, 7)$ and $q^9 = (12, 9, 9, 9, 9, 9)$.

4 Linear description of $Q(q;k)$

In this section we assume that q is integral and study the facets of the polyhedron $Q(q;k)$ defined in (5). The goal is to determine a complete and nonredundant linear description of this polyhedron. Initially, we study simple facets coming from the linear description of the polyhedron $P(q;k)$, before turning to the remaining facets of $Q(q;k)$.

Observe that $Q(q;k)$ is full dimensional since the $n+1$ points $x(q_k)$ and $x(q_k) - e_j$ for all $1 \le j \le n$ are in $Q(q;k)$ and are affinely independent. Moreover the n vectors $-e_j$ for all $1 \le j \le n$ are the extreme rays of $Q(q;k)$, implying that if an inequality $a^T x \le \alpha$ is facet defining for $Q(q;k)$ then $a \ge 0$. Note also that, due to the symmetry of $Q(q;k)$, each permutation $\tilde{a}$ of a yields a facet defining inequality $\tilde{a}^T x \le \alpha$. Hence, a complete description of $Q(q;k)$ may be obtained by considering all permutations of all facet defining inequalities $a^T x \le \alpha$ such that $a \ge 0$ and a is nonincreasing.

The next lemma concerns the question of strict inequality in the rearrangement inequality. Let $1 \le s_1 < \ldots < s_p \le n$, $s_0 = 0$ and $s_{p+1} = n+1$. A permutation π on the set N_n is called an *$(s_1, \dots, s_p)$-permutation* if $\pi(\{s_j, \dots, s_{j+1} - 1\}) = \{s_j, \dots, s_{j+1} - 1\}$ for $0 \le j \le p$. In other words, π defines a permutation on each of the "intervals" $\{s_j, \dots, s_{j+1} - 1\}$.

Lemma 4.1 *Let a and x be nonincreasing vectors in $\mathbf{R}^n$ and let x' be a permutation of x. Define $s_1, \ldots, s_p$ (uniquely) from the "levels" of a such that*

$$a_1 = \ldots = a_{s_1 - 1} > a_{s_1} = \ldots = a_{s_2 - 1} > \ldots > a_{s_p} = \ldots = a_n.$$

Then $a^T x' \leq a^T x$ and equality holds if and only if x' can be obtained by an $(s_1, \ldots, s_p)$-permutation of x.

Proof. Consider the relation for vectors with two components. We have $(a_1x_1 + a_2x_2) - (a_1x_2 + a_2x_1) = (a_1 - a_2)(x_1 - x_2)$. From this we see that (i) $a_1x_1 + a_2x_2 \geq a_1x_2 + a_2x_1$ if $x_1 \geq x_2$, and (ii) the inequality in (i) is strict if and only if $a_1 > a_2$ and $x_1 > x_2$. The desired result can be obtained from these observations by noting that x' is not a $(s_1, \ldots, s_p)$-permutation of x if and only if there exists indices i, j with $a_i > a_j$ and $x_i > x_j$. (Remark: the rearrangement inequality (1) follows similarly).

□

The next technical lemma will be helpful in the sequel for proving that a valid inequality is facet defining for $Q(q; k)$.

Lemma 4.2 *Let $a \geq 0$ be a nonincreasing vector in $\mathbf{R}^n$ such that $a^T x \leq \alpha$ is a valid inequality defining a nonempty face F of $Q(q; k)$. Let $1 \leq s \leq s' \leq t \leq n$ be such that $a_i = 0$ for all $t + 1 \leq i \leq n$ and $a_s = \ldots = a_{s'}$. Let $b^T x \leq \beta$ be a valid inequality defining a facet F' of $Q(q; k)$ with F contained in F'. Then*

(i) $b_i = 0$ for all $t + 1 \leq i \leq n$, and

(ii) if there exists a point y in F such that $y_i \neq y_j$ for $i, j \in \{s, \ldots, s'\}$ then $b_s = \ldots = b_{s'}$.

Proof. (i) Since F is nonempty, we can pick z in F. For all $t + 1 \leq j \leq n$, the point $z - e_j$ is also in F and F', implying that $0 = b^T(z - (z - e_j)) = b_j$ for all $t + 1 \leq j \leq n$.

(ii) Let y be in F such that $y_i \neq y_j$ for $i, j \in \{s, \ldots, s'\}$. Let $k, k' \in \{s, \ldots s'\}$, $k \neq k'$, let y^1 be obtained by permuting components i with k and j with k' in y, and let y^2 be obtained by permuting the components k and k' in y^1. By Lemma 4.1, all $(s, s' + 1)$-permutations of y are also in F, proving that both y^1 and y^2 are in F and therefore in F'. Thus $0 = b^T(y^1 - y^2) = (b_k - b_{k'})(y_i - y_j)$ and as $(y_i - y_j) \neq 0$, we have $b_k = b'_k$.

□

Proposition 4.3 *For each $1 \leq r \leq k$, the inequality*

$$\sum_{i=1}^{r} x_i \leq q_{1:r} \tag{10}$$

defines a facet of $Q(q; k)$ if and only if $r = 1$ or $q_1 > q_r$.

Proof. Let F be the face of $Q(q;k)$ induced by (10), written $a^T x \leq q_{1:r}$, and let $b^T x \leq \beta$ be an inequality inducing a facet F' of $Q(q;k)$ containing F. Note that $b \geq 0$ and we can assume w.l.o.g. that the smallest positive entry in b is equal to 1. By Lemma 4.2 (i), we have $b_i = 0$ for all $r+1 \leq i \leq n$.

Note that $x(q_k)$ is in F. Hence, if $q_1 > q_r$, Lemma 4.2 (ii) shows that $b_1 = \ldots = b_r$ and thus $a = b$. If $r = 1$ then we trivially have $a = b$. In both cases, since F is nonempty, we have $q_{1:r} = \beta$, i.e., (10) defines a facet of $Q(q;k)$.

Conversely, suppose that $r > 1$ and $q_1 = \ldots = q_r$. Then $a^T x \leq q_{1:r}$ is the sum of the valid inequalities $e_i^T x \leq q_1$ for all $1 \leq i \leq r$ and thus does not define a facet of $Q(q;k)$.

□

We call each inequality in (10) a *set size inequality.* In certain cases the set size inequalities give a complete linear description of $Q(q;k)$, or equivalently, $Q(q;k)$ and $P(q;k)$ coincide. In fact, from the characterization of the vertices of $P(q;k)$ and $Q(q;k)$ we see that this occurs precisely whenever all the q-averages w^s are integral, i.e., whenever $k-s | q_{s+1:k}$ for $s = 0, \ldots, s^*-1$. In general, however, further inequalities are required to give a complete linear description of $Q(q;k)$.

Let s and t be integers satisfying $0 \leq s < k < t \leq n$. Define $\delta^s = q_{s+1:k} - (k-s)\lfloor \bar{q}_{s+1:k} \rfloor$ which is the remainder modulo $k-s$ of $q_{s+1:k}$. Let $a^{s,t}$ be given by:

$$a_j^{s,t} = \begin{cases} (t-s-\delta^s)/(k-s-\delta^s) & \text{for } j = 1, \ldots, s, \\ 1 & \text{for } j = s+1, \ldots, t, \\ 0 & \text{for } j = t+1, \ldots, n \end{cases} \tag{11}$$

and $\alpha^{s,t} = ((t-k)/(k-s-\delta^s))q_{1:s} + q_{1:k} + (t-k)\lfloor \bar{q}_{s+1:k} \rfloor$. We call an inequality of the form $(a^{s,t})^T x \leq \alpha^{s,t}$ a *q-average inequality.* Note here that if $s > 0$ then $a_s^{s,t} > 1$ as $t > k$ and $\delta^s < k-s$. We call an inequality $b^T x \leq \alpha^{s,t}$ a *permuted q-average inequality* whenever b is a permutation of $a^{s,t}$. The following lemma gives a closed form for the optimum solution to the LP $\max\{(a^{s,t})^T x \mid x \in P(q;k)\}$ and shows that there are integral points in $Q(q;k)$ satisfying $(a^{s,t})^T x = \alpha^{s,t}$.

Lemma 4.4 *Let $0 \leq s < k < t \leq n$ and $m = \lfloor \bar{q}_{s+1:k} \rfloor$. Then*

(i) $(a^{s,t})^T w^s = \max\{(a^{s,t})^T x \mid x \in P(q;k)\}$,

(ii) $(a^{s,t})^T x(m) = \alpha^{s,t}$, *and*

(iii) if $m+1 \leq m^*$ then $(a^{s,t})^T x(m+1) = \alpha^{s,t}$.

Proof. To simplify the notation, let $a := a^{s,t}$. Due to Theorem 3.5 and the fact that $w^{s^*-1} = \ldots = w^{k-1}$, if $s < s^*-1$ then (i) is verified if $a^T w^s < a^T w^u$ for all $0 \leq u \leq s^*-1$, $u \neq s$ and if $s \geq s^*-1$ then (i) is verified if $a^T w^s < a^T w^u$ for all $0 \leq u < s^*-1$. Notice that, if $s \leq k-2$, then $a_{s+1:n}/(k-s) = (t-s)/(k-s) < (t-s-1)/(k-s-1) = a_{s+2:n}/(k-s-1)$ and Lemma 3.4 shows that $a^T w^s < a^T w^u$

for all $\min\{s, s^*-1\} < u \leq s^*-1$. If $s \geq 1$ then $a_s = (t-s-\delta^s)/(k-s-\delta^s) \geq (t-s)/(k-s)$ and thus $a_{s:n}/(k-s+1) \geq a_{s+1:n}/(k-s)$ as shown in Lemma 3.3. Thus Lemma 3.4 yields that $a^T w^s < a^T w^u$ for all $0 \leq u < \min\{s, s^*-1\}$ and (i) holds. Note that (ii) follows from

$$\begin{aligned}
&a^T x(m) = a_1 x(m)_{1:s} + x(m)_{s+1:k} + x(m)_{k+1:t} = \\
&(a_1 - 1)x(m)_{1:s} + x(m)_{1:k} + (t-k)m = \\
&((t-k)/(k-s-\delta^s))q_{1:s} + q_{1:k} + (t-k)\lfloor q_{s+1:k} \rfloor = \alpha^{s,t}
\end{aligned}$$

and, if $m+1 \leq m^*$ then $x(m+1)$ is defined and

$$\begin{aligned}
&a^T x(m+1) = a_1 x(m+1)_{1:s} + x(m+1)_{s+1:k} + x(m+1)_{k+1:t} = \\
&(a_1 - 1)(q_{1:k} - x(m+1)_{s+1:k}) + q_{1:k} + (t-k)(m+1) = \\
&(a_1 - 1)q_{1:s} + q_{1:k} + (t-k)m + (a_1 - 1)(q_{s+1:k} - x(m+1)_{s+1:k}) + \\
&(t-k) = \alpha^{s,t} + (a_1 - 1)(q_{s+1:k} - (k-s)(m+1)) + t - k = \\
&\alpha^{s,t} + (a_1 - 1)(\delta^s - (k-s)) + t - k = \alpha^{s,t}
\end{aligned}$$

as desired. □

We shall prove that all permuted q-average inequalities are valid for $Q(q;k)$. As a preparation for this we give relations between optimal solutions of LP problems over $P(q;k)$ and similar ones over $Q(q;k)$, and start with a result obtained from the last part of Lemma 3.2.

Lemma 4.5 *For each integral $q_k \leq m \leq \bar{q}_{1:k}$, $x(m)$ is a convex combination of $w^{s(m)}$ and $w^{s(m)+1}$.*

Proof. If $s(m) = k-1$ then, by definition of $x(m)$, we have $\Delta(m) = x(m)_k = q_k$ implying $x(m) = w^{k-1}$. Otherwise, by definition of $s(m)$, we have $\bar{q}_{s(m)+1:k} \geq m > \bar{q}_{s(m)+2:k}$ and the result follows from the last part of Lemma 3.2 with $j = s(m)+1$, $\alpha_1 = \bar{q}_{j:k}$ and $\alpha_2 = \bar{q}_{j+1:k}$. □

By the characterization of the extreme points of $P(q;k)$ given in Theorem 3.5 and of $Q(q;k)$ given in Proposition 3.7, for any $c \geq 0$ there exists $w \in W(q;k)$ and an integer α such that w and $x(\alpha)$ maximize $c^T x$ over $P(q;k)$ and $Q(q;k)$ respectively. The following proposition describes more precisely the relation between these optimal solutions when their values differ.

Proposition 4.6 *Let $c \geq 0$ be a nonincreasing vector in $\mathbf{R}^n$ such that*

$$c^T w^s = \max\{c^T x \mid x \in P(q;k)\} > \max\{c^T x \mid x \in Q(q;k)\} = c^T x(t). \tag{12}$$

Then t is either $\lfloor \bar{q}_{s+1:k} \rfloor$ or $\lfloor \bar{q}_{s+1:k} \rfloor + 1$.

Proof. We claim that for each $m \in \{m_*, \ldots, m^*\}$ the objective value $c^T x(m)$ is a convex combination of $c^T w^{s(m)}$ and $c^T w^{s(m)+1}$. To verify this, note that

$\bar{q}_{s(m)+2:k} < m \leq \bar{q}_{s(m)+1:k}$. Therefore, using Lemma 4.5, we see that $x(m)$ is a convex combination of the two adjacent q-averages $w^{s(m)}$ and $w^{s(m)+1}$. The claim follows due to the linearity of the objective function.

From Lemma 3.4 there are integers a and b with $0 \leq a \leq b \leq s^* - 1$ such that the ordering in (9) holds. From (12) it follows that $a \leq s \leq b$. Observe that $s(t) \notin \{a, \ldots, b-1\}$ for otherwise the claim would show that $c^T x(t) = c^T w^s$ contradicting the strict inequality in (12). Furthermore, combining the strict inequalities in (9) with the claim, we see that $c^T x(m)$ is maximized over m whenever m is either the floor or ceil of the (fractional) number $\bar{q}_{s+1:k}$ and the proof is complete.

□

The fact that each permuted q-average inequality is valid for $Q(q;k)$ is implied by the symmetry of $Q(q;k)$ and the following lemma:

Lemma 4.7 *Let $0 \leq s < k < t \leq n$ and $m := \lfloor \bar{q}_{s+1:k} \rfloor$. If $(a^{s,t})^T x \leq \alpha^{s,t}$ is not valid for $P(q;k)$, then*

(i) $(a^{s,t})^T x \leq \alpha^{s,t}$ is valid for $Q(q;k)$;

(ii) for all $m' \in \{m_*, \ldots, m^*\}$ we have $(a^{s,t})^T x(m') = \alpha^{s,t}$ if and only if $m' = m$ or $m' = m+1$;

(iii) An extreme point v of $Q(q;k)$ satisfies $(a^{s,t})^T v = \alpha^{s,t}$ only if v can be obtained by an $(s+1, t+1)$-permutation of $x(m)$ or $x(m+1)$.

Proof. To simplify the notation, let $a := a^{s,t}$. Lemma 4.4 shows that w^s is the optimum solution to $\max\{a^T x \mid x \in P(q;k)\}$. As $m = \lfloor w^s_k \rfloor$, Proposition 4.6 proves that $\max\{a^T x(t) \mid t \in \{m_*, \ldots, m^*\}\}$ is attained only for $t = m$ or $t = m+1$ or both, yielding (i) and (ii). By Lemma 4.1, if v is an extreme point of $Q(q;k)$ such that $a^T v = \alpha^{s,t}$ then the point v' obtained by sorting the components of v in nonincreasing order satisfies $a \cdot v' \geq a \cdot v$. Moreover equality holds if and only if v may be obtained by a $(s+1, t+1)$-permutation of v'. As v' is either $x(m)$ or $x(m+1)$, (iii) follows.

□

Consider the special case of the q-average inequalities obtained by setting $s = 0$; this leads to the inequality

$$\sum_{j=1}^{t} x_j \leq q_{1:k} + (t-k)\lfloor \bar{q}_{1:k} \rfloor \tag{13}$$

We call each such inequality an *extended set size inequality* since it "extends" the set size inequalities to sets of cardinality larger than k.

Example. Consider again our example where $k = 3$, $n = 5$ and $q = (7, 2, 1)$. We get the extended set size inequalities $x(\mathbf{N}_4) \leq 13$ and $x(\mathbf{N}_5) \leq 16$. Observe that both these inequalities cut off the fractional q-average $w^0 =$

$(10/3, \ldots, 10/3)$. Other q-average inequalities are $2x_1 + x_2 + x_3 + x_4 \leq 18$ (obtained for $s = 1$, $t = 4$) and $3x_1 + x_2 + x_3 + x_4 + x_5 \leq 26$ (for $s = 1$, $t = 5$).

We are now in position to show that each facet of $Q(q; k)$ that is not a facet of $P(q; k)$ is obtained from a permuted q-average inequality. Note that by scaling, one can assume that the smallest positive entry of a facet defining inequality is 1. This is implicit in the following proposition.

Proposition 4.8 *Let c be a nonincreasing vector in $\mathbf{R}^n$ and c_0 be a real number such that the inequality $c^T x \leq c_0$ defines a facet of $Q(q; k)$ and is not valid for $P(q; k)$. Then there exist $0 \leq s < k < t \leq n$ such that $c = a^{s,t}$ and $c_0 = \alpha^{s,t}$.*

Moreover for $h = \lfloor \bar{q}_{s+1:k} \rfloor$, (i) if $s > 0$ then $h + 1 \leq m^*$ and (ii) if $s > 1$ then either $q_1 > q_s$ or $h + 1 < \bar{q}_{1:k}$.

Proof. Let F be the facet of $Q(q; k)$ defined by $c^T x \leq c_0$ and $\mathcal{Q}$ be the set of nonincreasing extreme points of F. As $Q(q; k)$ is full dimensional, the face F' of $Q(q; k)$ defined by a valid inequality $(c')^T x \leq c'_0$ equals F if and only if there exists $\lambda > 0$ such that $c = \lambda c'$ and $c_0 = \lambda c'_0$. Moreover, if $\{1 \leq i \leq n \mid c_i = 0\} = \{1 \leq i \leq n \mid c'_i = 0\}$ then the extreme rays of F and F' are identical and thus $F = F'$ if and only if both faces have the same set of extreme points.

The rearrangement inequality (1) implies that if v is an extreme point of F then the vector obtained by sorting the components of v in nonincreasing order is in $\mathcal{Q}$ and each extreme points of F is obtained by a permutation of some vector in $\mathcal{Q}$. The assumptions imply that c satisfies the hypothesis of Proposition 4.6 and therefore $\mathcal{Q}$ contains at most two elements.

Let t be the largest index such that $c_t > 0$ and let t' be the smallest index such that $c_{t'} = c_t$.

Case 1: $t' = 1$. If $t \leq k$ then inequality $c^T x \leq c_0$ is of the form $\sum_{i=1}^{t} x_i \leq c_0$, implying $c_0 = q_{1:t}$ and this inequality is valid for $P(q; k)$, a contradiction. Otherwise, $t > k$ and thus $c = a^{s,t}$ with $s = 0$. Lemma 4.7 shows that $x(m^*)$ satisfies $(a^{0,t})^T x = \alpha^{0,t}$ and thus $c_0 = \alpha^{0,t}$.

Case 2: $t' > 1$. Let s be the largest index such that $c_1 = c_s$. Then $1 \leq s < t' \leq n$. Suppose that $|\mathcal{Q}| = 1$. Then Lemma 4.1 implies that $v_{1:s}$ has the same value for all extreme points v of F. Thus for small $\epsilon > 0$, if we define d by $d_i = c_i + \epsilon$ for all $i \leq s$, $d_i = c_i$ for all $s + 1 \leq i \leq n$, and $d_0 = c_0 + \epsilon \cdot v_{1:s}$, the inequality $(d)^T x \leq d_0$ is valid for $Q(q; k)$ and F is contained in the face of $Q(q; k)$ defined by this inequality, a contradiction. Hence $|\mathcal{Q}| = 2$ and Proposition 4.6 shows that there exists an integral m such that $\mathcal{Q} = \{x(m), x(m+1)\}$. It follows that $\bar{q}_{s+1:k} < m^*$ and (i) holds.

Case 2.1: $s < t' - 1$. If $x(m)_{1:s} = x(m + 1)_{1:s}$ then the same reasoning as in the case $|\mathcal{Q}| = 1$ yields a contradiction. If $x(m)_{t':t} = x(m + 1)_{t':t}$ then a similar reasoning (adding ϵ to the components $\{t', \ldots, t\}$ of c instead of the components $\{1, \ldots, s\}$) yields a contradiction. Let $\gamma^1 = x(m)_{1:s}$, $\delta^1 = x(m)_{t':t}$, $\gamma^2 = x(m + 1)_{1:s}$, $\delta^2 = x(m + 1)_{t':t}$, and ϵ, ϵ' such that $\epsilon/\epsilon' = (\delta^2 - \delta^1)/(\gamma^1 - \gamma^2)$.

For $\epsilon > 0$, define

$$c'_i = c_i + \epsilon \text{ for } 1 \le i \le s, \quad c'_i = c_i + \epsilon' \text{ for } t' \le i \le t, \quad c'_i = c_i \quad \text{otherwise} \tag{14}$$

and $c'_0 = c_0 + \epsilon \cdot \gamma^1 + \epsilon' \cdot \delta^1$. Then inequality $(c')^T x \le c'_0$ is valid for $Q(q;k)$ for small enough $\epsilon > 0$, as

$$(c')^T x(m) = c^T x(m) + \epsilon \cdot \gamma^1 + \epsilon' \cdot \delta^1 = c'_0$$

and

$$(c')^T(x(m) - x(m+1)) = c^T(x(m) - x(m+1)) + \epsilon \cdot (\gamma^1 - \gamma^2) + \epsilon' \cdot (\delta^1 - \delta^2) = 0 + \epsilon' \cdot (\delta^2 - \delta^1) + \epsilon' \cdot (\delta^1 - \delta^2) = 0.$$

Moreover, Lemma 4.1 shows that the face of $Q(q;k)$ defined by this inequality contains the facet F, a contradiction as (c', c'_0) is not a positive multiple of (c, c_0) due to the fact that $s < t' - 1$, implying $c_{t'-1} = c'_{t'-1} > 0$.

Case 2.2: $s = t' - 1$. Note that we have $t' \le k$ since otherwise $s \ge k$ and Lemma 3.4 shows that w^0 is the unique optimum solution to max $\{c^T x \mid x \in P(q;k)\}$. Then, by Proposition 4.6 and since $\lfloor \bar{q}_{1:k} \rfloor + 1 > m^*$, $\mathcal{Q}$ contains only the vector $x(m^*)$, a contradiction.

We also have $t > k$ since otherwise, by Lemma 3.4, w^{k-1} is optimal for max $\{c^T x \mid x \in P(q;k)\}$ and, as w^{k-1} is integral, it is also optimal for $\{c^T x \mid x \in Q(q;k)\}$. It follows that $c_0 = c^T w^{k-1}$, implying that inequality $c^T x \le c_0$ is valid for $P(q;k)$, a contradiction. Hence $1 \le s = t' - 1 < t' \le k < t \le n$.

We now show that $s(m+1) \le s - 1$ and $s(m) \ge s$, implying that $m = \lfloor \bar{q}_{s+1:k} \rfloor$. Indeed, if $s(m+1) > s-1$ then $s(m) > s-1$ and thus $x(m+1)_{1:s} = x(m)_{1:s} = q_{1:s}$. The same reasoning as in the case where $|\mathcal{Q}| = 1$ yields a contradiction. For the other inequality, observe that if $s(m) < s = t' - 1$ then $s(m+1) < t'-1$ and thus $x(m)_{t'} = \ldots = x(m)_t$ and $x(m+1)_{t'} = \ldots = x(m+1)_t$. Lemma 4.1 implies that this relation holds for all extreme points v of F and thus if we define d' by

$$d'_i = c_i \text{ for } i \ne t', t, \quad d'_{t'} = c_{t'} + \epsilon \quad \text{and} \quad d'_t = c_t - \epsilon, \tag{15}$$

all extreme points v of F satisfy $d'^T v = c_0$ for all $\epsilon > 0$. Moreover, $d'^T x \le c_0$ is valid for $Q(q;k)$ for $\epsilon > 0$ small enough, implying that the face of $Q(q;k)$ defined by this inequality contains F, a contradiction as d is not a positive multiple of c.

Note that the same reasoning proves that (ii) holds. Indeed, if (ii) does not hold then $q_1 = \ldots = q_s$, $m + 1 = \bar{q}_{1:k}$ and $s(m+1) = 0$, implying that $x(m)_1 = \ldots = x(m)_s$ and $x(m+1)_1 = \ldots = x(m+1)_s$.

By Lemma 4.1, a vector v is an extreme point of F only if it may be obtained by some $(s+1, t+1)$-permutation of either $x(m)$ or $x(m+1)$. Since $m = \lfloor \bar{q}_{s+1:k} \rfloor$, Lemma 4.7 shows that all these points are on the face of $Q(q;k)$ defined by ${a^{s,t}}^T x \le \alpha^{s,t}$, implying $c = a^{s,t}$ and $c_0 = \alpha^{s,t}$. □

Theorem 4.9 *A complete linear description of $Q(q;k)$ is given by the permuted set size inequalities and the permuted q-average inequalities.*

Proof. Proposition 4.8 implies that a facet F of $Q(q;k)$ is either a facet of $P(q;k)$ or there exist s,t such that F is defined by an inequality that is a permutation of $(a^{s,t})^T x \leq \alpha^{s,t}$. Therefore each facet of $Q(q;k)$ is either induced by a set size inequality or a permuted q-average inequality and the theorem follows.

□

Permuted q-average inequalities that are facet defining for $Q(q;k)$ are described in the next proposition.

Proposition 4.10 *Let $0 \leq s < k < t \leq n$ and $m = \lfloor \bar{q}_{s+1:k} \rfloor$. The inequality $(a^{s,t})^T x \leq \alpha^{s,t}$ defines a facet of $Q(q;k)$ if and only if*

(i) $\bar{q}_{s+1:k}$ is fractional,

(ii) if $s > 0$ then $m + 1 \leq m^*$, and

(iii) if $s > 1$ then either $q_1 > q_s$ or $m + 1 < \bar{q}_{1:k}$.

Proof. First, note that (i) is equivalent to saying that $(a^{s,t})^T x \leq \alpha^{s,t}$ is not valid for $P(q;k)$. Indeed, Lemma 4.4 shows that w^s is the optimum solution to max $\{(a^{s,t})^T x \mid x \in P(q;k)\}$ and that $x(m)$ satisfies $(a^{s,t})^T x(m) = \alpha^{s,t}$. If $\bar{q}_{s+1:k}$ is integral then $w^s = x(m)$ implying that the inequality is valid for $P(q;k)$. If $\bar{q}_{s+1:k}$ is fractional then since $w^s_{1:s} = q_{1:s} = x(m)_{1:s}$, $w^s_{s+1:k} = q_{s+1:k} = x(m)_{s+1:k}$ and $w^s_{k+1:t} > x(m)_{k+1:t}$, we have $(a^{s,t})^T (w^s - x(m)) = w^s_{k+1:t} - x(m)_{k+1:t} > 0$ and the inequality is not valid for $P(q;k)$.

Suppose that $(a^{s,t})^T x \leq \alpha^{s,t}$ defines a facet F of $Q(q;k)$. Then since F is not a facet of $P(q;k)$, this inequality is not valid for $P(q;k)$ and thus (i) holds. The result then follows from Proposition 4.8.

Suppose now that (i), (ii) and (iii) hold and let F be the face of $Q(q;k)$ defined by the inequality. Let F' be a facet of $Q(q;k)$ containing F and defined by $b^T x \leq \beta$. By Lemma 4.4, $x(m)$ is in F and Lemma 4.2 shows that $b_j = 0$ for all $t + 1 \leq j \leq n$.

We claim that $b_1 = \ldots = b_s$ and $b_{s+1} = \ldots = b_t$. To prove the claim, Lemma 4.2 shows that it suffices to find a point y in F such that $y_i \neq y_j$ for $i, j \in \{s+1, \ldots, t\}$ and a point z in F such that $z_i \neq z_j$ for $i, j \in \{1, \ldots, s\}$. Note that (i) implies $s(m+1) < s \leq s(m)$. If $s(m) = s$ then, due to (i), $x(m)_{s(m)+1} = \Delta(m) > m = x(m)_k$. If $s(m) \geq s + 1$ then $x(m)_t = m < q_{s(m)} = x(m)_{s(m)}$. In both cases, setting $y = x(m)$ yields that $b_{s+1} = \ldots = b_t$ and the second part of the claim follows. The first part of the claim is immediate if $s \leq 1$. Assume that $s > 1$. If $q_1 > q_s$ then $z = x(m)$ proves that $b_1 = \ldots = b_s$. Otherwise, note that (ii) implies that $x(m+1)$ exists and Lemma 4.4 shows that this point is in F. If $s(m+1) > 0$ then, by Lemma 3.1, $x(m+1)_{s(m)+1} = \Delta(m+1) <$

$q_{s(m+1)+1} \leq q_{s(m+1)} = x(m+1)_{s(m+1)}$. If $s(m+1) = 0$ then (iii) implies that $x(m+1)_1 = \Delta(m+1) > m+1 = x(m+1)_2$. In both cases $z = x(m+1)$ yields $b_1 = \ldots = b_s$, completing the proof of the claim.

Since $b \geq 0$, one can assume that the smallest positive entry of b equals 1. Hence, if $s = 0$ we have $a^{s,t} = b$. If $s > 0$, then, since (i) implies $q_{s+1} > m$, we have $x(m)_s = q_s \geq q_{s+1} > m = x(m)_k$ and the rearrangement inequality shows that $b_s \geq b_k$. It follows that b is nonincreasing and Proposition 4.8 proves that $a^{s,t} = b$. In both cases, since F is nonempty, we have $\alpha^{s,t} = \beta$ proving the proposition. □

For a discussion of simple algorithms for solving LP problems over $P(q;k)$ and $Q(q;k)$, see ([2]).

The vertices of $Q(q;k)$ can now be described.

Theorem 4.11 *The vertex set of $Q(q;k)$ consists of the vectors that can be obtained as permutation of the rounded q-average.*

Proof. Each vertex can be obtained as permutation of a rounded q-average as shown in Lemma 3.7. It remains to prove that $x(m)$ is indeed a vertex when m is q-extreme.

Let $x(m)$ be a rounded q-average. If $x(m)$ is a vertex of $P(q;k)$, then we are done as $Q(q;k) \subseteq P(q;k)$. Otherwise, by Proposition 4.10 there exists $0 \leq s < k < t \leq n$ such that $(a^{s,t})^T x(m) = \alpha^{s,t}$ and this inequality is not valid for $P(q;k)$. By Lemma 4.7, there exists at most one $m' \neq m$ such that $(a^{s,t})^T x(m') = \alpha^{s,t}$.

If $m < m'$ (or if m' does not exist), then $x(m)$ is lexicographically larger than $x(m')$ and thus for $\epsilon > 0$ small enough, $x(m)$ is the only optimal solution to max $\{\sum_{i=1}^n (a_i^{s,t} + \epsilon^i) x_i \mid x \in Q(q;k)\}$.

Otherwise, $x(m)$ is lexicographically smaller than $x(m')$ and thus a permutation of $x(m)$ is the only optimal solution to max $\{\sum_{i=1}^n (a_i^{s,t} + \epsilon^{n+1-i}) x_i \mid x \in Q(q;k)\}$.

□

5 Conclusions

We have studied the concept of weak k-majorization and associated polyhedra. Complete inner and outer descriptions were found for the k-majorization polyhedron $P(q;k)$ consisting of all vectors weakly k-majorized by a given vector as well as for the integer hull $Q(q;k)$ of $P(q;k)$. An interesting direction for further work is to study other polyhedra and optimization problems involving k-majorization. For instance, in some plant (facility) location problems or job assignment problems it may be of interest to consider additional k-majorization constraints. Both structural and algorithmic results would be of interest, and some work in this direction is ongoing.

References

[1] Dahl, G., *Pseudo experiments and majorization.* Thesis, University of Oslo, 1984.

[2] Dahl, G., Polyhedra and optimization in connection with a weak majorization ordering. Preprint 10, Dec. 1994, Institute of Informatics, University of Oslo. See also Springer Lecture Notes in Computer Science, Vol. 920, *Integer programming and combinatorial optimization*, (ed. E.Balas and J.Clausen), Springer, 1995.

[3] Edmonds, J., Matroids and the greedy algorithm, Mathematical Programming 1 (1971) 127-136.

[4] Fujishige, S. *Submodular functions and optimization*, Annals of Discrete Mathematics, Vol. 47. Amsterdam, North Holland, 1991.

[5] Grötschel, M., Lovász, L. and Schrijver, A., *Geometric algorithms and combinatorial optimization*, Springer, 1988.

[6] Hardy, G.H, Littlewood, J.E. and Polya, G., *Inequalities* (2. ed.), Cambridge Mathematical Library, Cambridge University Press, 1988.

[7] Marshall, A.W. and Olkin, I., *Inequalities: Theory of Majorization and Its Applications*, New York, Academic Press, 1979.

[8] Nemhauser, G.L. and Wolsey, L.A., *Integer programming and combinatorial optimization*, Wiley, 1988.

[9] Pulleyblank, W.R., "Polyhedral combinatorics," in Handbooks in Operations Research, Vol. 1, *Optimization*, ed. Nemhauser et al., North-Holland, 1989.

[10] Schrijver, A., *Theory of linear and integer programming*, Wiley, 1986.

[11] Torgersen, E., *Comparison of Statistical Experiments*, Cambridge University Press, Cambridge, 1992.