The Circuit Polytope

Latife Genç-Kaya   J. N. Hooker

Carnegie Mellon University

Abstract

The circuit constraint requires that a sequence of n vertices in a directed graph describe a hamiltonian cycle. The constraint is useful for the succinct formulation of sequencing problems, such as the traveling salesman problem. We analyze the circuit polytope as an alternative to the traveling salesman polytope as a means of obtaining linear relaxations. We show that the facet-defining inequalities containing any given subset of at most $n - 4$ variables are precisely those defined by affinely independent sets of undominated circuits. We show further that a greedy algorithm generates all the undominated circuits. This leads to fast generation of sparse facets with separation heuristics. Finally, we show that proper choice of the numerical values that index the vertices can allow the resulting relaxation to exploit structure in the objective function.

Key words: circuit constraint, global constraint, polyhedral theory, traveling salesman problem

Email addresses: lgenc@andrew.cmu.edu (Latife Genç-Kaya), john@hooker.tepper.cmu.edu (J. N. Hooker).

1 The Circuit Constraint

The *circuit constraint* requires that a sequence of vertices in a directed graph define a hamiltonian circuit.

Let G be a directed graph on vertices $1, \ldots, n$, and let variable x_i denote the vertex that follows vertex i in the sequence. The *domain* D_i of each variable x_i (i.e, the set of values x_i can take) is the set of integers j for which (i, j) is an edge of G. The constraint

$$\text{circuit}(x_1, \ldots, x_n) \tag{1}$$

requires that $x = (x_1, \ldots, x_n)$ describe a hamiltonian circuit of G. For brevity, we will say that an x satisfying (1) is a *circuit*.

More precisely, x is a circuit if $\pi_1, \ldots, \pi_n$ is a permutation of $1, \ldots, n$, where $\pi_1 = 1$ and $\pi_{i+1} = x_{\pi_i}$ for $i = 1, \ldots, n-1$. Thus $\pi_1, \ldots, \pi_n$ indicates the order in which the vertices are visited. For example, if $\{1, 2, 3\}$ is the domain of each variable x_i, then $(x_1, x_2, x_3) = (3, 1, 2)$ is a circuit because $(\pi_1, \pi_2, \pi_3) = (1, 3, 2)$ is a permutation. The circuit goes from 1 to 3 to 2, and back to 1. However, $(x_1, x_2, x_3) = (1, 2, 3)$ is not a circuit, because $(\pi_1, \pi_2, \pi_3) = (1, 1, 1)$ is not a permutation.

If x is a circuit, the sequence $x_1, \ldots, x_n$ is itself a permutation, but a given permutation x need not be a circuit. In fact, if the domain of each x_i is $\{1, \ldots, n\}$, then $n!$ values of x are permutations but only $(n-1)!$ of these are circuits. In the above example, there are six permutations but only two circuits, namely $(2, 3, 1)$ and $(3, 1, 2)$.

The circuit constraint is useful for formulating combinatorial problems that involve permutations or sequencing. One of the best known such problems is the traveling salesman problem, which may be very succinctly written

$$
\begin{aligned}
&\min \sum_{i=1}^{n} c_{ix_i} \\
&\text{circuit}(x_1, \ldots, x_n), \quad x_i \in D_i, \ i = 1, \ldots n
\end{aligned}
\tag{2}
$$

where c_{ij} is the distance from city i to city j. The objective is to visit each city once, and return to the starting city, in such a way as to minimize the total travel distance.

Domain filtering methods for the circuit constraint appear in [2,6,9]. These can be useful for eliminating infeasible values from the variable domains. The object of the present paper is to study the circuit polytope, so as to obtain a relaxation for the circuit constraint that can be combined with filtering to accelerate solution further.

2 The Circuit Polytope

The *circuit polytope* is the convex hull of the feasible solutions of (1) when G is a complete graph; that is, when each variable domain D_i is $\{1, \ldots, n\}$. To our knowledge, this polytope has not been studied. Rather, the circuit constraint is generally formulated by replacing the variables x_i with 0-1 variables y_{ij}, where $y_{ij} = 1$ if vertex j immediately follows vertex i in the hamiltonian circuit. The traveling salesman problem (2), for example, is typically written

$$
\begin{aligned}
&\min \sum_{ij} c_{ij} y_{ij} \\
&\sum_j y_{ij} = \sum_j y_{ji} = 1, \quad i = 1, \dots, n && (a) \\
&\sum_{\substack{i \in V \\ j \notin V}} y_{ij} \geq 1, \quad \text{all } V \subset \{1, \dots, n\} \text{ with } 2 \leq |V| \leq n-2 && (b) \\
&y_{ij} \in \{0, 1\}, \quad \text{all } i, j && (c)
\end{aligned}
\tag{3}
$$

The polyhedral structure of problem (3) has been intensively analyzed, and surveys of this work may be found in [1,5,7].

Rather than introduce 0-1 variables, we analyze the circuit polytope directly. We show that all facets can be identified by by generating *undominated* circuits. A subset of these facet-defining inequalities can be assembled to obtain a tight continuous relaxation of the circuit constraint.

This approach has four possible advantages. (a) The facet-defining inequalities are expressed in terms of n variables, rather than n^2 variables as in the conventional approach. (b) The inequalities are quite different from the traditional traveling salesman cuts and may have complementary strengths. (c) Because the variables can take arbitrary values (not just $1, \dots, n$), these values can be chosen to exploit structure in the objective function coefficients. (d) We can describe the circuit polytope with a degree of generality that does not appear to be possible for the 0-1 traveling salesman polytope.

We have not demonstrated these advantages computationally. The goal of this paper is to lay the theoretical groundwork by describing the circuit polytope, which is an interesting object of study in its own right.

3 Arbitrary Domains

A peculiar characteristic of the circuit constraint is that the values of its variables are indices of other variables. Because the vertex immediately after x_i is x_{x_i}, the value of x_i must index a variable. The numbers $1, \ldots, n$ are normally used as indices, but this is an arbitrary choice. One could just as well use any other set of distinct numbers, which would give rise to a different circuit polytope. Thus the circuit polytope cannot be fully understood unless it is characterized for general numerical domains, and not just for $1, \ldots, n$. This also provides more modeling flexibility that can be used to exploit problem structure (Section 11).

We therefore generalize the circuit constraint so that each domain D_i is drawn from an arbitrary set $\{v_1, \ldots, v_n\}$ of nonnegative real numbers. The constraint is written

$$\text{circuit}(x_{v_1}, \ldots, x_{v_n}) \tag{4}$$

It is convenient to assume $v_1 < \cdots < v_n$. Thus $\text{circuit}(x_0, x_{2.3}, x_{3.1})$ is a well-formed circuit constraint if the variable domains are subsets of $\{0, 2.3, 3.1\}$. The nonnegativity of the v_is does not sacrifice generality, since one can always translate the origin so that the feasible points lie in the nonnegative orthant.

To avoid an additional layer of subscripts, we will consistently abuse notation by writing x_{v_i} as x_i. We therefore write the constraint (4) as

$$\text{circuit}(x_1, \ldots, x_n) \tag{5}$$

Thus $x = (x_1, \ldots, x_n)$ satisfies (5) if and only if $\pi_1, \ldots, \pi_n$ is a permutation of $1, \ldots, n$, where $\pi_1 = 1$ and $v_{\pi_i} = x_{\pi_{i-1}}$ for $i = 2, \ldots n$.

We define the circuit polytope $C_n(v)$ with respect to $v = (v_1, \ldots, v_n)$ to be the convex hull of the feasible solutions of (5) for full domains; that is, each domain D_i is $\{v_1, \ldots, v_n\}$. All of the facet-defining inequalities we identify below for full domains are valid inequalities for smaller domains, even if they may not define facets of the convex hull.

The circuit polytope has a different character than most polytopes studied in combinatorial optimization. Normally the shape of the polytope does not depend on particular numerical values, but only on the structure of the problem. Because the structure of the circuit polytope depends on the domain values, the polytope is partly a discrete and partly a continuous object. This will be reflected in combinatorial and numerical phases of the method for generating facets.

4 Overview of the Results

Our main result is that facet-defining inequalities for the circuit polytope are precisely the valid inequalities defined by sets of undominated circuits. More precisely, the facet-defining inequalities containing any selected subset of at most $n-4$ variables are defined by sets of circuits that are undominated with respect to those variables. A circuit is undominated with respect to the selected variables if no other circuit assigns smaller or equal values to these variables. The result is refined to take account of the signs of terms containing the variables.

A second key result is that the undominated circuits can be generated with a greedy algorithm. The algorithm has exponential complexity in general but is fast when only a few variables are selected. This leads to a separation heuristic that is quite fast when restricted to sparse facets. It is through separation heuristics that useful inequalities are normally obtained in practice.

We begin by establishing the dimension of the circuit polytope (Theorem 1). We then prove the main result stated above (Theorems 4 and 5). This allows us to identify all facet-defining inequalities containing a given selection of at most $n-4$ variables, if we can generate undominated circuits with respect to the selected variables.

We then describe a greedy algorithm that generates these circuits. This is a purely combinatorial problem that does not depend on the particular domain values $v_1, \ldots, v_n$. Strictly speaking, the algorithm generates partial circuits by specifying values only for the selected variables. We prove that it finds all undominated partial circuits (Theorems 7 and 8).

We can now identify facet-defining inequalities containing a given subset of variables by solving a continuous, numerical problem. We compute the inequalities defined by affinely independent sets of undominated partial circuits. We check which inequalities are satisfied by the remaining partial circuits, given the particular numerical values of the domain elements. The inequalities that pass this test are facet-defining. The procedure is fast when the facets are sparse (contain a small number of variables).

At this point we observe some consequences of our results. We first contrast the circuit polytope with the permutation polytope, which contains the circuit polytope, and whose facial structure is well known. We identify a large class of permutation facets that are also circuit facets (Corollary 9). The circuit polytope is more complicated than the permutation polytope, however, and unlike the permutation polytope, its structure depends on the domain values. We also explicitly identify all two-term facets of the circuit polytope (Corollary 10).

We then address the separation problem, which is the problem of identifying facet-defining inequalities that are violated by a solution of the current relaxation of the problem. We describe two separation heuristics, one of which seeks separating inequalities with all positive coefficients, and one which seeks inequalities with arbitrary coefficients. The heuristics are fast because they begin by seeking the sparsest facets.

We conclude by showing how knowledge of the circuit polytope for arbitrary domains can allow one to exploit cost structure in the objective function of the problem.

5 Dimension of the Polytope

We begin by establishing the dimension of the circuit polytope.

Theorem 1 *The dimension of the circuit polytope $C_n(v)$ is $n-2$ for $n = 2, 3$ and $n-1$ for $n \geq 4$.*

Proof. The polytope $C_n(v)$ is a point (v_2, v_1) for $n = 2$ and the line segment from (v_2, v_3, v_1) to (v_3, v_1, v_2) for $n = 3$. In either case the dimension is $n - 2$.

To prove the theorem for $n \geq 4$, note first that all feasible points for (5) satisfy

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} v_i \tag{6}$$

(Recall that x_i is shorthand for x_{v_i}.) Thus, $C_n(v)$ has dimension at most $n-1$. To show it has dimension exactly $n-1$, it suffices to exhibit n affinely independent points in $C_n(v)$. Consider the following n permutations of $v_1, \ldots, v_n$, where the first $n-1$ permutations consist of v_1 followed by cyclic permutations of $v_2, \ldots, v_n$. The last permutation is obtained by swapping v_{n-1} and v_n in the first permutation:

$$\begin{array}{lllllll} v_1, & v_2, & v_3, & \ldots, & v_{n-2}, & v_{n-1}, & v_n \\ v_1, & v_3, & v_4, & \ldots, & v_{n-1}, & v_n, & v_2 \\ v_1, & v_4, & v_5, & \ldots, & v_n, & v_2, & v_3 \\ & & & \vdots & & & \\ v_1, & v_{n-1}, & v_n, & \ldots, & v_{n-4}, & v_{n-3}, & v_{n-2} \\ v_1, & v_n, & v_2, & \ldots, & v_{n-3}, & v_{n-2}, & v_{n-1} \\ v_1, & v_2, & v_3, & \ldots, & v_{n-2}, & v_n, & v_{n-1} \end{array} \tag{7}$$

The rows of the following matrix correspond to circuit representations of the above permutations. Thus row i contains the values $x_1, \ldots, x_n$ for the ith permutation in (7).

$$\begin{bmatrix} v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n & v_1 \\ v_3 & v_1 & v_4 & \cdots & v_{n-1} & v_n & v_2 \\ v_4 & v_3 & v_1 & \cdots & v_{n-1} & v_n & v_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-1} & v_3 & v_4 & \cdots & v_1 & v_n & v_2 \\ v_n & v_3 & v_4 & \cdots & v_{n-1} & v_1 & v_2 \\ v_2 & v_3 & v_4 & \cdots & v_n & v_1 & v_{n-1} \end{bmatrix} \tag{8}$$

Since each row of (8) is a point in $C_n(v)$, it suffices to show that the rows are affinely independent. Subtract $[v_n \;\; v_3 \;\; v_4 \;\; \cdots \;\; v_{n-1} \;\; v_n \;\; v_2]$ from every row of (8) to obtain

$$
\begin{bmatrix}
v_2 - v_n & 0 & 0 & \cdots & 0 & 0 & v_1 - v_2 \\
v_3 - v_n & v_1 - v_3 & 0 & \cdots & 0 & 0 & 0 \\
v_4 - v_n & 0 & v_1 - v_4 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
v_{n-1} - v_n & 0 & 0 & \cdots & v_1 - v_{n-1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & v_1 - v_n & 0 \\
v_2 - v_n & 0 & 0 & \cdots & v_n - v_{n-1} & v_1 - v_n & v_{n-1} - v_2
\end{bmatrix} \tag{9}
$$

The rows of (8) are affinely independent if and only if the rows of (9) are. It now suffices to show that (9) is nonsingular, and we do so through a series of row operations. The first step is to subtract $(v_{n-1} - v_2)/(v_1 - v_2)$ times row 1, $(v_n - v_{n-1})/(v_1 - v_{n-1})$ times row $n-2$, and row $n-1$ from row n to obtain

$$
\begin{bmatrix}
v_2 - v_n & 0 & 0 & \cdots & 0 & 0 & v_1 - v_2 \\
v_3 - v_n & v_1 - v_3 & 0 & \cdots & 0 & 0 & 0 \\
v_4 - v_n & 0 & v_1 - v_4 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
v_{n-1} - v_n & 0 & 0 & \cdots & v_1 - v_{n-1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & v_1 - v_n & 0 \\
E_n & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix} \tag{10}
$$

where

$$
E_n = -\frac{v_n - v_{n-1}}{v_{n-1} - v_1}(v_n - v_{n-1}) - \frac{v_{n-1} - v_1}{v_2 - v_1}(v_n - v_2)
$$

Interchange the first and last rows of (10) to obtain

$$\begin{bmatrix} E_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ v_1 - v_n & v_1 - v_3 & 0 & \cdots & 0 & 0 & 0 \\ v_4 - v_n & 0 & v_1 - v_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ v_{n-1} - v_n & 0 & 0 & \cdots & v_1 - v_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_1 - v_n & 0 \\ v_2 - v_n & 0 & 0 & \cdots & 0 & 0 & v_1 - v_2 \end{bmatrix} \tag{11}$$

Note that $E_n < 0$ since $v_1 < \cdots < v_n$. Thus (11) is a lower triangular matrix with nonzero diagonal elements and is therefore nonsingular. □

6 Facets of the Polytope

We now describe facets of the circuit polytope $C_n(v)$. The following lemma is key.

Lemma 2 *Suppose that the inequality*

$$\sum_{j \in J} a_j x_j \geq \alpha \tag{12}$$

is valid for circuit$(x_1, \ldots, x_n)$ *and is satisfied as an equation by at least one circuit* x. *If* $|J| \leq n - 4$ *and*

$$\sum_{j=1}^{n} d_j x_j = \delta \tag{13}$$

is satisfied by all circuits x *that satisfy (12) as an equation, then* $d_i = d_j$ *for all* $i, j \notin J$.

Proof. Because $|J| \leq n - 4$, it suffices to prove that $d_{j_1} = d_{j_2} = d_{j_3} = d_{j_4}$ for any four distinct indices $j_1, \ldots, j_4 \notin J$.

Let x^0 be any circuit that satisfies (12) as an equation, and let the permutation described by x^0 be

$$v_1, \ldots, v_{j_1-1}, v_{j_1}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_3+1}, \ldots, v_{j_4-1}, v_{j_4}$$

Consider the circuits $x^1, \ldots, x^5$ that describe the following permutations, respectively:

$$v_1, \ldots, v_{j_1-1}, v_{j_1}, v_{j_3+1}, \ldots, v_{j_4-1}, v_{j_4}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}$$
$$v_1, \ldots, v_{j_1-1}, v_{j_1}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_3+1}, \ldots, v_{j_4-1}, v_{j_4}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}$$
$$v_1, \ldots, v_{j_1-1}, v_{j_1}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_3+1}, \ldots, v_{j_4-1}, v_{j_4}$$
$$v_1, \ldots, v_{j_1-1}, v_{j_1}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}, v_{j_3+1}, \ldots, v_{j_4-1}, v_{j_4}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}$$
$$v_1, \ldots, v_{j_1-1}, v_{j_1}, v_{j_3+1}, \ldots, v_{j_4-1}, v_{j_4}, v_{j_2+1}, \ldots, v_{j_3-1}, v_{j_3}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_2}$$

We obtain $x^1, \ldots, x^5$ from x^0 by viewing the permutation represented by the latter as a concatenation of four subsequences, each ending in one of the values v_{j_i}. We fix the first subsequence and obtain x^1 and x^2 by cyclically permuting the remaining three subsequences. We obtain x^3, x^4 and x^5 by interchanging a pair of subsequences.

Note that variables $x_{j_1}, \ldots, x_{j_4}$ have the values shown below in each circuit x^i:

x_{j_1}	x_{j_2}	x_{j_3}	x_{j_4}	
v_{j_1+1}	v_{j_2+1}	v_{j_3+1}	v_1	(x^0)
v_{j_3+1}	v_{j_2+1}	v_1	v_{j_1+1}	(x^1)
v_{j_2+1}	v_1	v_{j_3+1}	v_{j_1+1}	(x^2)
v_{j_2+1}	v_{j_3+1}	v_{j_1+1}	v_1	(x^3)
v_{j_1+1}	v_{j_3+1}	v_1	v_{j_2+1}	(x^4)
v_{j_3+1}	v_1	v_{j_1+1}	v_{j_2+1}	(x^5)

and all other variables x_j have value x_j^0 in each circuit x^i. Thus all six circuits $x^0, \ldots, x^5$ satisfy (12) as an equation, so that $dx^i = \delta$ for $i = 0, \ldots, 5$. This implies

$$\frac{1}{2}\begin{bmatrix} (dx^0+dx^1+dx^5)-(dx^2+dx^3+dx^4) \\ (dx^0+dx^2+dx^5)-(dx^1+dx^3+dx^4) \\ (dx^0+dx^3+dx^5)-(dx^1+dx^2+dx^4) \\ (dx^0+dx^2+dx^4)-(dx^1+dx^3+dx^5) \\ (dx^0+dx^4+dx^5)-(dx^1+dx^2+dx^3) \\ (dx^0+dx^1+dx^3)-(dx^2+dx^4+dx^5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting the values of $x^0,\ldots,x^5$, we obtain

$$\begin{bmatrix} v_{j_3+1}-v_{j_2+1} & v_{j_2+1}-v_{j_3+1} & 0 & 0 \\ 0 & v_1-v_{j_3+1} & v_{j_3+1}-v_1 & 0 \\ 0 & 0 & v_{j_1+1}-v_1 & v_1-v_{j_1+1} \\ v_{j_1+1}-v_{j_3+1} & 0 & v_{j_3+1}-v_{j_1+1} & 0 \\ v_{j_1+1}-v_{j_2+1} & 0 & 0 & v_{j_2+1}-v_{j_1+1} \\ 0 & v_{j_2+1}-v_1 & 0 & v_1-v_{j_2+1} \end{bmatrix} \begin{bmatrix} d_{j_1} \\ d_{j_2} \\ d_{j_3} \\ d_{j_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from which we can conclude that $d_{j_1} = d_{j_2} = d_{j_3} = d_{j_4}$. □

Lemma 2 applies only when $|J| \le n-4$ because its proof relies on the absence of at least four variables from (12). Theorems 4 and 6 are therefore stated only for $|J| \le n-4$. We conjecture that the theorems also hold for the densest facets ($|J| > n-4$), but proof seems to require the analysis of several special cases that substantially complicate the argument. This slightly stronger result would add little practical value in any case, because the separation algorithm is designed to generate sparse facets.

For a given x, it will be convenient to denote by $x(J)$ the tuple $(x_{j_1},\ldots,x_{j_m})$ when $J = \{j_1,\ldots,j_m\}$. We say that $x(J)$ is a *J-circuit* if it creates no cycles and is therefore a partial solution of the circuit constraint. That is, $x(J)$

is a J-circuit if there is no subsequence $j_{i_1}, \ldots, j_{i_k}$ of the indices in J such that $j_{i_{t+1}} = x_{j_{i_t}}$ for $t = 1, \ldots, k-1$ and $j_{i_1} = x_{j_{i_k}}$. The following lemma is straightforward, but its proof introduces notation we will need later.

Lemma 3 *If $\bar{x}(J)$ is a J-circuit, then there is a circuit x such that $x(J) = \bar{x}(J)$.*

Proof. Let $J = \{j_1, \ldots, j_m\}$, and let $\{v_{i_1}, \ldots, v_{i_r}\}$ be the subset of domain values $v_1, \ldots, v_n$ that occur in neither $\{v_{j_1}, \ldots, v_{j_m}\}$ nor $\{\bar{x}_{j_1}, \ldots, \bar{x}_{j_m}\}$. Consider the directed graph $G_{\bar{x}(J)}$ that contains a vertex v_i for each $i \in \{1, \ldots, n\}$, a directed edge $(v_{j_k}, \bar{x}_{j_k})$ for $k = 1, \ldots, m$, and a directed edge $(v_{i_k}, v_{i_{k+1}})$ for each $k = 1, \ldots, r-1$. The maximal subchains of $G_{\bar{x}(J)}$ have the form

$$
\begin{array}{l}
v_{j_{k_1}} \to \cdots \to v_{j_{k'_1}} \to \bar{x}_{j_{k'_1}} \\
v_{j_{k_2}} \to \cdots \to v_{j_{k'_2}} \to \bar{x}_{j_{k'_2}} \\
\vdots \\
v_{j_{k_p}} \to \cdots \to v_{j_{k'_p}} \to \bar{x}_{j_{k'_p}} \\
v_{i_1} \to \cdots \to v_{i_r}
\end{array}
$$

Because maximal subchains are disjoint, we can form a hamiltonian circuit in $G_{\bar{x}(J)}$ by linking the last element of each subchain to the first element of the next, and linking v_{i_r} to v_{k_1}. Let $v_{s_1}, \ldots, v_{s_n}$ be the resulting circuit. Then if x is given by $x_i = v_{s_{((i-1) \bmod n)+1}}$ for $i = 1, \ldots, n$, then x is a circuit and $x(J) = \bar{x}(J)$. $\square$

The idea of domination is central to characterizing facets of $C_n(v)$. Let $J = J_+ \cup J_-$ (with $J_+ \cap J_- = \emptyset$) be a subset of variable indices. For $i \in J$ we say that $x_i \preceq y_i$ if $i \in J_+$ and $x_i \leq y_i$, or $i \in J_-$ and $x_i \geq y_i$. Also $x_i \prec y_i$ if $x_i \preceq y_i$ and $x_i \neq y_i$. We say that $x'(J)$ *dominates* $x(J)$ with respect to $J = J_+ \cup J_-$ when $x'_j \preceq x_j$ for all $j \in J$. A J-circuit $x(J)$ is *undominated* with respect to $J = J_+ \cup J_-$ if no other J-circuit dominates it with respect to $J = J_+ \cup J_-$.

The following theorem provides the basis for generating facets of $C_n(v)$ by generating undominated J-circuits.

Theorem 4 *Consider any inequality of the form (12). Let S be the set of J-circuits that are undominated with respect to $J = J_+ \cup J_-$, where $J_+ = \{j \mid a_j > 0\}$, $J_- = \{j \mid a_j < 0\}$, and $1 \leq |J| \leq n-4$. Let S' be any subset of $|J|$ affinely independent J-circuits in S. If the J-circuits in S' satisfy*

$$\sum_{j \in J} a_j x_j = \alpha \tag{14}$$

and the J-circuits in $S \setminus S'$ satisfy (12), then (12) defines a facet of $C_n(v)$.

Proof. Let $S = \{x^1(J), \ldots, x^m(J)\}$. We first show that (12) is valid; that is, satisfied by any circuit. Thus let x be any circuit. Because S contains all undominated J-circuits, $x(J)$ is dominated by some $x^i(J) \in S$ with respect to $J = J_+ \cup J_-$, which means that $a_j(x_j - x^i_j) \geq 0$ for all $j \in J$. Thus we have

$$\sum_{j \in J} a_j x_j \geq \sum_{j \in J} a_j x^i_j \geq \alpha$$

because $x^i(J)$ satisfies (12), and so x satisfies (12).

Now let (13) be any equation satisfied by all circuits x that satisfy (12) as an equation. Recall that all circuits satisfy (6). Thus to show (12) is facet-defining, it suffices to show that (13) is a linear combination of (14) and (6).

Because $|J| \geq 1$ and S is therefore nonempty, at least one J-circuit $x^i(J) \in S$ satisfies (12) as an equation. Lemma 3 therefore implies that at least one circuit x^i satisfies (12) as an equation. Thus since $|J| \leq n-4$, we have from Lemma 2 that $d_i = d_j$ for all $i, j \notin J$.

We first suppose that $d_j = 0$ for all $j \notin J$. Then (13) has the form

$$\sum_{j \in J} d_j x_j = \delta \tag{15}$$

Because the J-circuits in S' are affinely independent and satisfy (14) and (15), these two equations are the same up to a scalar multiple. Thus (13) is a linear combination of (14) and (6), where the latter has multiplier zero.

We now suppose that $d_j \neq 0$ for $j \notin J$. Because the d_js are equal for all $j \notin J$, we can without loss of generality write (13) as

$$\sum_{j \in J} d_j x_j + \sum_{j \notin J} x_j = \delta$$

This is a linear combination of (14) and (6) if the following is a scalar multiple of (14):

$$\sum_{j \in J} (d_j - 1) x_j = \delta - \sum_{j=1}^{n} v_j \tag{16}$$

But this follows from the fact that the J-circuits in S' are affinely independent and satisfy (14) and (16). □

We show now that the previous theorem completely characterizes facet-defining inequalities having at most $n - 4$ terms.

Theorem 5 *Consider any inequality (12) that is facet-defining for a circuit polytope $C_n(v)$, and let $J_+ = \{j \mid a_j > 0\}$ and $J_- = \{j \mid a_j < 0\}$. Then there are affinely independent J-circuits $x^1(J), \ldots, x^{|J|}(J)$ that are undominated with respect to $J = J_+ \cup J_-$ and satisfy (14).*

Proof. Any facet-defining inequality (12) is satisfied as an equation by n affinely independent circuits $\bar{x}^1, \ldots, \bar{x}^n$. Then $\{\bar{x}^1(J), \ldots, \bar{x}^n(J)\}$ has some subset $\{\bar{x}^{j_1}(J), \ldots, \bar{x}^{j_{|J|}}(J)\}$ of $|J|$ affinely independent J-circuits. These are undominated with respect to $J = J_+ \cup J_-$, because otherwise, some J-circuit

$\hat{x}(J)$ strictly dominates some $\bar{x}^{j_i}(J)$ with respect to $J = J_+ \cup J_-$. Also by Lemma 3, $\hat{x}(J)$ is part of some circuit $\hat{x}$. This means

$$\sum_{j \in J} a_j \hat{x}_j < \sum_{j \in J} a_j \bar{x}_j^{j_i} = \alpha$$

and $\hat{x}$ violates (12). This implies that (12) is not valid and therefore is not facet-defining as assumed. □

We sum up the foregoing results as follows.

Theorem 6 *Consider any inequality for the form (12). Let S be the set of J-circuits that are undominated with respect to $J = J_+ \cup J_-$, where $J_+ = \{j \mid a_j > 0\}$ and $J_- = \{j \mid a_j < 0\}$. If $1 \leq |J| \leq n - 4$, then (12) is facet-defining for $C_n(v)$ if and only if all circuits in S satisfy (12) and S contains $|J|$ affinely independent circuits satisfying (12) as an equation.*

7 Facet Generation

The results of the previous section indicate how to generate all facet-defining inequalities for $C_n(v)$ having at most $n - 4$ terms.

Suppose we wish to generate all such facet-defining inequalities that contain a given set of variables with a given sign pattern. That is, we are given $J = J_+ \cup J_-$ and wish to generate all facet-defining inequalities (12) with $a_j > 0$ for $j \in J_+$ and $a_j < 0$ for $j \in J_-$. We first use the greedy algorithm of the next section to generate the set S of all J-circuits that are undominated with respect to $J = J_+ \cup J_-$. We then consider all affinely independent subsets of $|J|$ J-circuits in S. Each subset uniquely defines an equation (14) up to a scalar multiple. If the remaining J-circuits in S satisfy (12), then we list (12) as a facet-defining inequality.

Note that we do not identify a facet by generating n affinely independent circuits that define the facet, as this would be a much more difficult task in general. Rather, we generate $|J|$ affinely independent J-circuits that define the coefficients of an inequality containing $|J|$ terms. This inequality defines a facet if it is valid, which we can easily check.

As an example, consider

$$\text{circuit}(x_1, \ldots, x_7) \tag{17}$$

where each x_i has domain $\{v_1, \ldots, v_7\} = \{2, 5, 6, 7, 9, 10, 12\}$. By Theorem 1, the corresponding polytope $C_7(2, 5, 6, 7, 9, 10, 12)$ has dimension 6. Its affine hull is described by $x_1 + \cdots + x_7 = 51$. Suppose that we wish to identify all facet-defining inequalities of the form

$$a_1x_1 + a_3x_3 + a_4x_4 \geq \alpha, \quad \text{with } a_1, a_3, a_4 > 0 \tag{18}$$

Four J-circuits $\bar{x}^i(J)$ are undominated with respect to $J = J_+ = \{1, 3, 4\}$. They are independent of the particular domain values $v_1, \ldots, v_7$ and can therefore be written

$$\begin{aligned} \bar{x}^1(J) &= (v_2, v_1, v_3) \\ \bar{x}^2(J) &= (v_2, v_4, v_1) \\ \bar{x}^3(J) &= (v_3, v_2, v_1) \\ \bar{x}^4(J) &= (v_4, v_1, v_2) \end{aligned} \tag{19}$$

(We will show how to obtain these J-circuits using a greedy algorithm in the next section.) There are four subsets of three J-circuits ($|J| = 3$), shown in Table 1, and each uniquely defines a hyperplane and a corresponding inequal-

Table 1
Hyperplanes determined by undominated J-circuits for example (17).

Defining J-circuits	Uniquely defined hyperplane $a(J)x(J) = \alpha$	Is $a(J)x(J) \geq \alpha$ valid?
$\bar{x}^1(J), \bar{x}^2(J), \bar{x}^3(J)$	$8x_1 + 4x_3 + 5x_4 = 78$	Yes, satisfied by $\bar{x}^4(J)$
$\bar{x}^1(J), \bar{x}^2(J), \bar{x}^4(J)$	$5x_1 + 8x_3 + 10x_4 = 101$	No, violated by $\bar{x}^3(J)$
$\bar{x}^1(J), \bar{x}^3(J), \bar{x}^4(J)$	$3x_1 + 7x_3 + 6x_4 = 65$	Yes, satisfied by $\bar{x}^2(J)$
$\bar{x}^2(J), \bar{x}^3(J), \bar{x}^4(J)$	$6x_1 + 3x_3 + x_4 = 53$	No, violated by $\bar{x}^1(J)$

ity. The first inequality, defined by $\bar{x}^1(J)$, $\bar{x}^2(J)$, and $\bar{x}^3(J)$, is satisfied by the remaining J-circuit $\bar{x}^4(J)$, and similarly for the third inequality. The second and fourth inequalities, however, are violated by the remaining J-circuit and are not valid. This means there are exactly two facets defined by inequalities of the form (18), namely those defined by

$$8x_1 + 4x_3 + 5x_4 \geq 78$$
$$3x_1 + 7x_3 + 6x_4 \geq 65$$

Now we find all facet-defining inequalities of the form (18) but with $a_1, a_3 > 0$ and $a_4 < 0$, so that $J_+ = \{1, 3\}$ and $J_- = \{4\}$. In this case, there is only one undominated circuit, $x(J) = (v_2, v_1, v_7)$. Because we do not have three undominated circuits to define a hyperplane, there are no facet-defining inequalities of this form.

8 Generation of Undominated Circuits

A simple greedy procedure can be used to generate all J-circuits $\bar{x}(J)$ that are undominated with respect to $J = J_+ \cup J_-$. It is applied for each ordering $j_1, \ldots, j_m$ of the elements of J. First, let $\bar{x}_{j_1}$ be the smallest domain value v_i

For each ordering $j_1, \ldots, j_m$ of the elements of J:
  Let $\bar{J} = \{1, \ldots, n\}$ and $J' = \emptyset$.
  For $i = 1, \ldots, m$:
    Add j_i to J'.
    If $j_i \in J_+$ then let $\bar{x}_{j_i}$ be the minimum value v_k in $\{v_i \mid i \in \bar{J}\}$
      such that $\bar{x}(J')$ is a J'-circuit.
    Else let $\bar{x}_{j_i}$ be the maximum value v_k in $\{v_i \mid i \in \bar{J}\}$
      such that $\bar{x}(J')$ is a J'-circuit.
    Remove k from $\bar{J}$.
  Add $\bar{x}(J)$ to the list of undominated J-circuits.

Fig. 1. Greedy procedure for generating undominated J-circuits. Input: tuple v of domain values and $J = J_+ \cup J_-$. Output: a complete list of J-circuits that are undominated with respect to $J = J_+ \cup J_-$.

if $j_1 \in J_+$, or the largest if $j_1 \in J_-$. Then let $\bar{x}_{j_2}$ be the smallest (or largest) remaining domain value that does not create a cycle. Continue until all $\bar{x}_j$ for $j \in J$ are defined. The precise algorithm appears in Fig. 1.

Theorem 7 *The greedy procedure of Fig. 1 generates J-circuits that are undominated with respect to $J = J_+ \cup J_-$.*

Proof. Let $\bar{x}(J)$ be a J-circuit generated by the procedure for a given ordering $j_1, \ldots, j_m$. To see that $\bar{x}(J)$ is undominated with respect to $J = J_+ \cup J_-$, assume otherwise. Then there exists a J-circuit $\bar{y}(J)$ such that $\bar{x}(J) \succeq \bar{y}(J)$ and $\bar{x}_{j_t} \succ \bar{y}_{j_t}$ for some $t \in \{1, \ldots, m\}$. Let t be the smallest such index, so that $\bar{x}_{j_k} = \bar{y}_{j_k}$ for $k = 1, \ldots, t-1$. This contradicts the greedy construction of $\bar{x}$, because $\bar{y}_{j_t}$ is available when $\bar{x}_{j_t}$ is assigned to x_{j_t}. □

For example, the undominated circuits (19) for circuit constraint (17) can be generated by considering the six orderings of $J = J_+ = \{1, 3, 4\}$, listed on the left below. The resulting undominated J-circuits appear on the right.

$1, 3, 4 \qquad (v_2, v_1, v_3) = \bar{x}^1(J)$

$1, 4, 3 \qquad (v_2, v_4, v_1) = \bar{x}^2(J)$

$3, 1, 4 \qquad (v_2, v_1, v_3) = \bar{x}^1(J)$

$3, 4, 1 \qquad (v_4, v_1, v_2) = \bar{x}^4(J)$

$4, 1, 3 \qquad (v_2, v_4, v_1) = \bar{x}^2(J)$

$4, 3, 1 \qquad (v_3, v_2, v_1) = \bar{x}^3(J)$

When $J_+ = \{1, 3\}$ and $J_- = \{4\}$, all six orderings result in the same J-circuit (v_2, v_1, v_7).

The greedy algorithm of Fig. 1 enumerates $m!$ permutations of the m elements of J. For each permutation it sorts at most n elements m times, for a worst-case compleity of $\mathcal{O}(m!mn \log n)$. The complexity is therefore exponential, but exponential only in the size of $|J|$, not the overall number n of variables. It is therefore practical to generate all facet-defining inequalities for a small J, as is done in the separation algorithms of Section 10.

It remains to prove that the greedy procedure finds all undominated J-circuits. To do so, we define for any given circuit $\bar{x}$ an *implied ordering* with respect to $J = J_+ \cup J_-$. The proof will show that if $\bar{x}$ is undominated with respect to $J = J_+ \cup J_-$, then a J-circuit that is greedily constructed according to the implied ordering is identical to $\bar{x}(J)$.

For a given circuit $\bar{x}$ and $J = J_+ \cup J_-$, let $|J| = m$, $J_+ = \{i_1, \ldots, i_p\}$, and $J_- = \{j_1, \ldots, j_q\}$. The contents of J_+ are ordered so that $\bar{x}_{i_1} < \cdots < \bar{x}_{i_p}$, and the contents of J_- are ordered so that $\bar{x}_{j_1} > \cdots > \bar{x}_{j_q}$.

The implied ordering will be $k_1, \ldots, k_m$. As we construct the ordering, we construct a J-circuit $y(J)$ that is greedy with respect to the ordering. The basic idea is that at each step ℓ of the procedure, we assign the greedy value

Let $V = \{v_1, \ldots, v_n\}$.
Let $J_+ = \{i_1, \ldots, i_p\}$ where $\bar{x}_{i_1} < \cdots < \bar{x}_{i_p}$.
Let $J_- = \{j_1, \ldots, j_q\}$ where $\bar{x}_{j_1} > \cdots > \bar{x}_{j_q}$.
Let $r = 1$ and $s = 1$.
For $\ell = 1, \ldots, m$:
    Let $v_{\min}$ be the smallest value in V such that setting $y_{i_r} = v_{\min}$
        creates no cycle with the elements of y assigned so far.
    Let $v_{\max}$ be the largest value in V such that setting $y_{j_s} = v_{\max}$
        creates no cycle with the elements of y assigned so far.
    If $r \le p$ and ($\bar{x}_{i_r} = v_{\min}$ or $s > q$) then
        Let $k_\ell = i_r$, $y_{i_r} = v_{\min}$, and $r = r + 1$.
        Remove $v_{\min}$ from V.
    Else
        Let $k_\ell = j_s$, $y_{j_s} = v_{\max}$, and $s = s + 1$.
        Remove $v_{\max}$ from V.

Fig. 2. Algorithm for generating an implied ordering $k_1, \ldots, k_m$ for circuit $\bar{x}$ with respect to $J = J_+ \cup J_-$, where $m = |J|$. The resulting J-circuit $y(J)$ is greedily constructed with respect to the ordering $k_1, \ldots, k_m$ and $J = J_+ \cup J_-$. The algorithm is used to help prove Theorem 8, not to identify undominated J-circuits or construct facets.

to y_{i_r} for the next $i_r \in J_+$ (if any remain) and let $k_\ell = i_r$, provided this assigns y_{i_r} the same value as $\bar{x}_{i_r}$. Otherwise, we assign the greedy value to y_{j_s} for the next $j_s \in J_-$ and let $k_\ell = j_s$. If no indices j_s remain in J_-, we assign the greedy value to y_{i_r} regardless of whether it agrees with $\bar{x}_{i_r}$. The precise algorithm appears in Fig. 2.

As an example, suppose $\bar{x} = (v_2, v_3, v_4, v_7, v_6, v_1, v_5)$, $J_+ = \{1, 3, 6, 7\}$, and $J_- = \{4, 5\}$. Thus $\bar{x}(J) = (\bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6, \bar{x}_7) = (v_2, v_4, v_7, v_6, v_1, v_5)$. Based on the values in $\bar{x}(J)$, we order the contents of J_+ so that $J_+ = \{i_1, \ldots, i_4\} = \{6, 1, 3, 7\}$. Similarly, $J_- = \{j_1, j_2\} = \{4, 5\}$. The progress of the algorithm appears in Table 2. Note that when $\ell = 4$, we first consider assigning $v_{\min}$ to y_{i_r}. But this results in $y_7 = v_3$, which deviates from $\bar{x}$ because $\bar{x}_7 = v_5$. We therefore assign $v_{\max}$ to y_{j_s}, which yields $y_4 = v_7$. When $\ell = 5$, we again consider assigning $v_{\min}$ to y_{i_r}, but because $v_{\min}$ has changed, we now obtain an assignment $y_7 = v_5$ that agrees with $\bar{x}$. When $\ell = 6$, the indices in J_+ are ex-

Table 2
Computation of the implied ordering for $\bar{x} = (v_2, v_3, v_4, v_7, v_6, v_1, v_5)$, where $J_+ = \{1, 3, 6, 7\}$ and $J_- = \{4, 5\}$ (indicated by the the signs above $\bar{x}$).

								$+$		$+$	$-$	$-$	$+$	$+$	
							$\bar{x} =$	v_2	v_3	v_4	v_7	v_6	v_1	v_5	
ℓ	r	s	i_r	j_s	$v_{\min}$	$v_{\max}$		y_1	y_2	y_3	y_4	y_5	y_6	y_7	k_ℓ
1	1	1	6	4	v_1	v_7							v_1		6
2	2	1	1	4	v_2	v_7		v_2					v_1		1
3	3	1	3	4	v_4	v_7		v_2		v_4			v_1		3
4	4	1	7	4	v_3	v_7		v_2		v_4	v_7		v_1		4
5	4	2	7	5	v_5	v_6		v_2		v_4	v_7		v_1	v_5	5
6	5	2		5		v_6		v_2		v_4	v_7	v_6	v_1	v_5	7

hausted, and we therefore assign $v_{\min}$ to y_{j_s}, so that $y_5 = v_6$. The resulting $y(J)$ is identical to $\bar{x}(J)$, and the implied ordering is $(k_1, \ldots, k_6) = (6, 1, 3, 4, 5, 7)$.

Theorem 8 *Any undominated circuit with respect to $J = J_+ \cup J_-$ can be generated in a greedy fashion for some ordering of the indices in J.*

Proof. Let $\bar{x}$ be a circuit that is undominated with respect to $J = J_+ \cup J_-$, where $|J| = m$, $J_+ = \{i_1, \ldots, i_p\}$ and $J_- = \{j_1, \ldots, j_q\}$. Suppose the contents of J_+ are ordered so that $\bar{x}_{i_1} < \cdots < \bar{x}_{i_p}$, and the contents of J_- are ordered so that $\bar{x}_{j_1} > \cdots > \bar{x}_{j_q}$.

Let $k_1, \ldots, k_m$ be the implied ordering for $\bar{x}$ with respect to $J = J_+ \cup J_-$ as computed above, and let $(y_{k_1}, \ldots, y_{k_m})$ be the greedy solution with respect to this ordering. We claim that $\bar{x}_{k_\ell} = y_{k_\ell}$ for $\ell = 1, \ldots, m$, which suffices to prove the theorem. Supposing to the contrary, let $\bar{\ell}$ be the smallest index for which $\bar{x}_{k_{\bar{\ell}}} \neq y_{k_{\bar{\ell}}}$. Clearly $\bar{x}_{k_{\bar{\ell}}} \prec y_{k_{\bar{\ell}}}$ is inconsistent with the greedy choice, because $\bar{x}_{k_{\bar{\ell}}}$ is available when $y_{k_{\bar{\ell}}}$ is assigned a value. Thus we have $\bar{x}_{k_{\bar{\ell}}} \succ y_{k_{\bar{\ell}}}$

By hypothesis, $\bar{x}$ is undominated with respect to $J = J_+ \cup J_-$. We therefore have $\bar{x}_{k_\ell} \prec y_{k_\ell}$ for some $\ell \in \{\bar{\ell}+1, \ldots, m\}$. Let $\hat{\ell}$ be the smallest such index. Then there are two cases: (1) $k_{\bar{\ell}}$ and $k_{\hat{\ell}}$ are both in J_+ or both in J_-, or (2) they are in different sets.

Case 1: $k_{\bar{\ell}}$ and $k_{\hat{\ell}}$ are both in J_+ or both in J_-. We will suppose that both are in J_+. The argument is similar if both are in J_-.

Let t be the index such that $i_t = k_{\bar{\ell}}$, and u the index such that $i_u = k_{\hat{\ell}}$. Then $\bar{x}_{i_t} > y_{i_t}$ because $\bar{x}_{i_t} \succ y_{i_t}$ and $i_t \in J_+$. Let t' be the largest index in $\{t, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > y_{i_{t'}}$. We know that t' exists because $\bar{x}_{i_t} > y_{i_t}$. Thus we have two sequences of values related as follows:

$$\begin{array}{ccccccccccccccccc}
\bar{x}_{i_1} & < \cdots < & \bar{x}_{i_{t-1}} & < & \bar{x}_{i_t} & < \cdots < & \bar{x}_{i_{t'-1}} & < & \bar{x}_{i_{t'}} & < \cdots < & \bar{x}_{i_{u-1}} & < & \bar{x}_{i_u} \\
= & & = & & > & & \geq & & > & & \geq & & < \\
y_{i_1} & \cdots & y_{i_{t-1}} & & y_{i_t} & \cdots & y_{i_{t'-1}} & & y_{i_{t'}} & \cdots & y_{i_{u-1}} & & y_{i_u}
\end{array}$$

Let u' be the largest index for which $y_{j_{u'}}$ has been assigned a value at the time y_{i_u} is assigned a value. We have the two sequences of values

$$\begin{array}{ccccc}
\bar{x}_{j_1} & > \cdots > & \bar{x}_{j_{u'-1}} & > & \bar{x}_{j_{u'}} \\
y_{j_1} & \cdots & y_{j_{u'-1}} & & y_{j_{u'}}
\end{array}$$

We first show that value $\bar{x}_{i_u}$ has not yet been assigned in the greedy algorithm when y_{i_u} is assigned a value. That is, we show that $\bar{x}_{i_u} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$ and $\bar{x}_{i_u} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$. To see that $\bar{x}_{i_u} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_u} = y_{i_w}$ for some $w \in \{1, \ldots, u-1\}$. This is impossible, because $\bar{x}_{i_u} > \bar{x}_{i_w} \geq y_{i_w}$. Also $\bar{x}_{i_u} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$, because assigning value $\bar{x}_{i_u}$ to y_{j_w} for some $w \in \{1, \ldots, u'\}$ contradicts the greedy construction of y, due to the fact that value y_{i_u} was available at that time and is a superior choice.

We next show that value $\bar{x}_{i_{t'}}$ has not yet been assigned in the greedy algorithm when y_{i_u} is assigned a value. That is, we show that $\bar{x}_{i_{t'}} \notin \{y_{i_1}, \ldots, y_{i_{u-1}}\}$ and $\bar{x}_{i_{t'}} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$. To begin with, we have that $\bar{x}_{i_{t'}} \notin \{y_{i_1}, \ldots, y_{i_{t'-1}}\}$, by virtue of the same reasoning just applied to $\bar{x}_{i_u}$. Also $\bar{x}_{i_{t'}} \neq y_{i_{t'}}$, since by hypothesis $\bar{x}_{i_{t'}} > y_{i_{t'}}$. To show that $\bar{x}_{i_{t'}} \notin \{y_{i_{t'+1}}, \ldots, y_{i_{u-1}}\}$, suppose to the contrary that $\bar{x}_{i_{t'}} = y_{i_w}$ for some $w \in \{t'+1, \ldots, u-1\}$. Then since $\bar{x}_{i_{t'}} < \bar{x}_{i_w}$, we must have $\bar{x}_{i_w} > y_{i_w}$. But this contradicts the definition of t' $(< w)$ as the largest index in $\{1, \ldots, u-1\}$ such that $\bar{x}_{i_{t'}} > y_{i_{t'}}$. Thus $\bar{x}_{i_{t'}} \neq y_{i_w}$. Finally, $\bar{x}_{i_{t'}} \notin \{y_{j_1}, \ldots, y_{j_{u'}}\}$ because assigning value $\bar{x}_{i_{t'}}$ to y_{j_w} for some $w \in \{1, \ldots, u'\}$ contradicts the greedy construction of y, due to the fact that y_{i_u} was available at the time and $y_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$.

Because $\bar{x}_{i_u} < y_{i_u}$ and value $\bar{x}_{i_u}$ has not yet been assigned, setting $y_{i_u} = \bar{x}_{i_u}$ must create a cycle in y, because otherwise setting $y_{i_u} = \bar{x}_{i_u}$ would have been the greedy choice. Also, setting $y_{i_u} = \bar{x}_{i_{t'}}$ was not the greedy choice because $y_{i_u} > \bar{x}_{i_u} > \bar{x}_{i_{t'}}$. Thus setting $y_{i_u} = \bar{x}_{i_{t'}}$ must likewise create a cycle in y, because $\bar{x}_{i_{t'}}$ has not yet been assigned. Now define $G_{y(J)}$ as before and consider the maximal subchain in $G_{y(J)}$ that contains y_{i_u}. Let the segment of the subchain up to y_{i_u} be

$$v_z \rightarrow \cdots \rightarrow v_{i_u} \rightarrow y_{i_u}$$

Because setting $y_{i_u} = \bar{x}_{i_u}$ creates a cycle in y, we must have $\bar{x}_{i_u} = v_z$. Similarly, because setting $y_{i_u} = \bar{x}_{i_{t'}}$ creates a cycle in y, we must have $\bar{x}_{i_{t'}} = v_z$. This implies $\bar{x}_{i_u} = \bar{x}_{i_{t'}}$, which is impossible because $\bar{x}_{i_u} > \bar{x}_{i_{t'}}$.

Case 2: $k_{\bar{\ell}} \in J_+$ and $k_{\hat{\ell}} \in J_-$, or $k_{\bar{\ell}} \in J_-$ and $k_{\hat{\ell}} \in J_+$. We can rule out the latter subcase immediately, because $k_{\bar{\ell}}$ can be in J_- only if $r > p$ when $y_{k_{\bar{\ell}}}$ is assigned a value. This means $k_{\hat{\ell}}$ must be in J_- as well, because $y_{k_{\hat{\ell}}}$ is assigned a value after $y_{k_{\bar{\ell}}}$ is assigned a value, and the situation reverts to Case 1. We therefore suppose $k_{\bar{\ell}} \in J_+$ and $k_{\hat{\ell}} \in J_-$.

Let t be the index such that $i_t = k_{\bar{\ell}}$, and u the index such that $j_u = k_{\hat{\ell}}$. Again $\bar{x}_{i_t} > y_{i_t}$ because $\bar{x}_{i_t} \succ y_{i_t}$ and $j_t \in J_+$. Thus, at the time value y_{i_t} was assigned a value, we had $\bar{x}_{j_s} < v_{\max}$ for the current value of s. So we have two sequences of values related as follows:

$$\begin{array}{ccccccccc} \bar{x}_{j_1} > \cdots > & \bar{x}_{j_{s-1}} > & \bar{x}_{j_s} > & \cdots & \bar{x}_{j_{u-1}} > & \bar{x}_{j_u} \\ = & = & \leq & & \leq & > \\ y_{j_1} \quad \cdots & y_{j_{s-1}} & y_{j_s} & \cdots & y_{j_{u-1}} & y_{j_u} \end{array} \tag{20}$$

where $v_{\max} > \bar{x}_{j_s}$. Let t' be the largest index for which $y_{i_{t'}}$ has been assigned a value at the time y_{j_u} is assigned a value. We have two sequences of values related as follows:

$$\begin{array}{cccccc} \bar{x}_{i_1} < \cdots < & \bar{x}_{i_{t-1}} < & \bar{x}_{i_t} < & \cdots < & \bar{x}_{i_{t'}} \\ = & = & > & & \\ y_{i_1} \quad \cdots & y_{i_{t-1}} & y_{i_t} & \cdots & y_{i_{t'}} \end{array}$$

We first show that a cycle must be created if value $\bar{x}_{j_u}$ is assigned to y_{j_u}. Because $y_{j_u} < \bar{x}_{j_u}$, it suffices to show that value $\bar{x}_{j_u}$ has not yet been assigned in the greedy algorithm when y_{j_u} is assigned a value. That is, we show that $\bar{x}_{j_u} \notin \{y_{j_1}, \ldots, y_{j_{u-1}}\}$ and $\bar{x}_{j_u} \notin \{y_{i_1}, \ldots, y_{i_{t'}}\}$. If $\bar{x}_{j_u} = y_{j_w}$ for some $w \in \{1, \ldots, u-1\}$, then $\bar{x}_{j_u} < \bar{x}_{j_w} \leq y_{j_w}$, which is impossible. Thus $\bar{x}_{j_u} \notin \{y_{j_1}, \ldots, y_{j_{u-1}}\}$. Also $\bar{x}_{j_u} \notin \{y_{i_1}, \ldots, y_{i_{t'}}\}$, because assigning value $\bar{x}_{j_u}$ to y_{i_w} for some $w \in \{1, \ldots, t'\}$ contradicts the greedy construction of y, due to the fact that value y_{j_u} was available at that time and is a superior choice.

We next show that a cycle must be created if value $v_{\max}$ is assigned to y_{j_u}. Note that $v_{\max} \notin \{y_{i_1}, \ldots, y_{i_{t'}}\}$, because assigning value $v_{\max}$ to y_{i_w} for some $w \in \{1, \ldots, t'\}$ contradicts the greedy construction of y, due to the fact that value y_{j_u} was available at that time and is a superior choice because $v_{\max} > \bar{x}_{j_s} > \bar{x}_{j_u}$. Now suppose, contrary to the claim, that assigning $v_{\max}$ to y_{j_u} does not create a cycle. Then since $v_{\max} > y_{j_u}$, the value $v_{\max}$ must have already been assigned in the greedy algorithm at the time y_{j_u} is assigned a value. This implies $v_{\max} \in \{y_{j_s}, \ldots, y_{j_{u-1}}\}$. But in this case we must have $y_{j_s} = v_{\max}$, because assigning $v_{\max}$ to y_{j_s} does not create a cycle and, by definition, is the most attractive choice at the time. Thus (20) becomes

$$\begin{array}{ccccccccccccc}
\bar{x}_{j_1} & > \cdots > & \bar{x}_{j_{s-1}} & > & \bar{x}_{j_s} & > \cdots > & \bar{x}_{j_{s'-1}} & > & \bar{x}_{j_{s'}} & > \cdots > & \bar{x}_{j_{u-1}} & > & \bar{x}_{j_u} \\
= & & = & & < & & \leq & & < & & \geq & & > \\
y_{j_1} & \cdots & y_{j_{s-1}} & & y_{j_s} & \cdots & y_{j_{s'-1}} & & y_{j_{s'}} & \cdots & y_{j_{u-1}} & & y_{j_u}
\end{array}$$

where $y_{j_s} = v_{\max}$ and where s' is the largest index in $\{s, \ldots, u-1\}$ such that $y_{j_{s'}} < \bar{x}_{j_{s'}}$. Now we can argue as in Case 1 that assigning $\bar{x}_{j_u}$ to y_{j_u} creates a cycle, and assigning $\bar{x}_{j_{s'}}$ to y_{j_u} creates a cycle, which implies $\bar{x}_{j_{s'}} = \bar{x}_{j_u}$, a contradiction because $\bar{x}_{j_{s'}} > \bar{x}_{j_u}$. We conclude that assigning $v_{\max}$ to y_{j_u} creates a cycle.

Having shown that assigning $\bar{x}_{j_u}$ to y_{j_u} creates a cycle, and assigning $v_{\max}$ to y_{j_u} creates a cycle, we derive as in Case 1 that $v_{\max} = \bar{x}_{j_u}$, a contradiction because $v_{\max} \geq \bar{x}_{j_s} > \bar{x}_{j_u}$. The theorem follows. □.

9 Permutation and Two-term Facets

In this section we examine two special classes of facets of $C_n(v)$—permutation facets and two-term facets.

The *permutation polytope* or *permutohedron* has been studied for at least a century [8]. The permutation polytope $P_n(v)$ for an arbitrary domain $\{v_1, \ldots, v_n\}$

can be defined as the convex hull of all points whose coordinates are permutations of $v_1, \ldots, v_n$. The circuit polytope $C_n(v)$ is contained in $P_n(v)$ because every circuit $(x_1, \ldots, x_n)$ is a permutation of $v_1, \ldots, v_n$. This means that every facet-defining inequality for $P_n(v)$ is valid for circuit but not necessarily facet defining. This raises the question as to which permutation facets are also circuit facets. We will identify a large family of permutation facets that can be immediately recognized as circuit facets.

The permutation polytope $P_n(v)$ has dimension $n-1$. The facets of $P_n(v)$ are identified in [3,10], and they are defined by

$$\sum_{j \in J} x_j \geq \sum_{j=1}^{|J|} v_j \tag{21}$$

for all $J \subset \{1, \ldots, n\}$ with $1 \leq |J| \leq n-1$. (Recall that $v_1 < \cdots < v_n$.) This result is generalized in [4] to domains with more than n elements.

For example, the permutation polytope $P_3(v)$ with $v = (2, 4, 5)$ is defined by

$$\begin{aligned}
&x_1 + x_2 + x_3 = 11 \\
&x_i \geq 2, \text{ for } i = 1, 2, 3 \\
&x_i + x_j \geq 6, \text{ for distinct } i, j \in \{1, 2, 3\}
\end{aligned}$$

We can see at this point that a facet-defining inequality for $P_n(v)$ need not be facet-defining for $C_n(v)$. The inequality $x_1 + x_2 \geq 6$ is facet-defining for $P_3(v)$ but not for $C_3(v)$, which is the line segment from $(4, 5, 2)$ to $(5, 2, 4)$.

Theorems 4, 7, and 8 allow us to identify a family of permutation facets that are also circuit facets.

Corollary 9 *The inequality (21) defines a facet of* $C_n(v)$ *if* $1 \leq |J| \leq n-4$ *and* $j \geq |J|$ *for all* $j \in J$.

Proof. Let $J = \{j_1, \ldots, j_m\}$. Due to Theorem 7 and the fact that $j \geq m$ for all $j \in J$, the following are undominated J-circuits with respect to $J = J_+$:

$$\text{all } \bar{x}(J) \text{ for which } \bar{x}_{j_1}, \ldots, \bar{x}_{j_m} \text{ is a permutation of } v_1, \ldots, v_m \tag{22}$$

Theorem 8 tells us that (22) is the complete set of J-circuits that are undominated with respect to $J = J_+$. Consider the following J-circuits from (22):

$$\begin{aligned}
\bar{x}^1(J) &= (v_1, v_2, v_3, v_4, \ldots, v_{m-1}, v_m) \\
\bar{x}^2(J) &= (v_2, v_1, v_3, v_4, \ldots, v_{m-1}, v_m) \\
\bar{x}^3(J) &= (v_1, v_3, v_2, v_4, \ldots, v_{m-1}, v_m) \\
&\vdots \\
\bar{x}^m(J) &= (v_1, v_2, v_3, v_4, \ldots, v_m, v_{m-1})
\end{aligned} \tag{23}$$

where $\bar{x}^i(J)$ is obtained for $i > 1$ by swapping v_{i-1} and v_i in $\bar{x}^1(J)$. These circuits are affinely independent, as can be seen by subtracting $\bar{x}^1(J)$ from each. By construction, all the J-circuits (23) satisfy (21) as an equation. Thus the affinely independent J-circuits (22) satisfy (21) as an equation, and all the remaining J-circuits in (23) satisfy (21). So by Theorem 4, (21) is facet-defining. □

We can check on a case-by-case basis whether permutation facets other than those mentioned in Corollary 9 are circuit facets. For example, if $J = J_+ = \{3, 4, 5\}$, then application of the greedy procedure in Fig. 1 yields the undominated J-circuits

$$\begin{aligned}
\bar{x}^1 &= (v_1, v_2, v_3) & \bar{x}^4 &= (v_4, v_1, v_2) \\
\bar{x}^2 &= (v_1, v_3, v_2) & \bar{x}^5 &= (v_2, v_3, v_1) \\
\bar{x}^3 &= (v_2, v_1, v_3) & \bar{x}^6 &= (v_4, v_2, v_1)
\end{aligned}$$

Some subsets of three J-circuits, such as $\{\bar{x}^1, \bar{x}^2, \bar{x}^3\}$, satisfy (21) as an equation. Because the remaining J-circuits clearly satisfy (21), the permutation facet (21) is also a circuit facet.

Another special class of facet-defining inequalities are those containing two terms. Because a set of two undominated J-circuits (where $|J| = 2$) defines exactly one facet, the two-term facets can be exhaustively listed in closed form.

Corollary 10 *If $n \geq 6$, the two-term facets of $C_n(v)$ are precisely those defined by*

$$\begin{aligned}
&x_i + x_j \geq v_1 + v_2, \quad \textit{for distinct } i, j \in \{3, \ldots, n\} \\
&(v_3 - v_1)x_1 + (v_3 - v_2)x_2 \geq v_3^2 - v_1 v_2 \\
&(v_2 - v_1)x_2 + (v_3 - v_1)x_i \geq v_2 v_3 - v_1^2, \quad \textit{for } i \in \{3, \ldots, n\} \\
&x_i + x_j \leq v_{n-1} + v_n, \quad \textit{for distinct } i, j \in \{1, \ldots, n-2\} \\
&(v_{n-1} - v_{n-2})x_{n-1} + (v_n - v_{n-2})x_n \leq v_n v_{n-1} - v_{n-2}^2 \\
&(v_n - v_{n-2})x_i + (v_n - v_{n-1})x_{n-1} \leq v_n^2 - v_{n-1} v_{n-2}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } i \in \{1, \ldots, n-2\}
\end{aligned}$$

The proof is straightforward.

10 Separation Heuristics

The greedy procedure described above for generating undominated J-circuits suggests some simple separation heuristics. Suppose we have a solution $\hat{x}$ of the current relaxation of the problem, and that $\hat{x}$ violates the circuit constraint. The *separation problem* is to find one or more facet-defining inequalities that separate $\hat{x}$ from the circuit polytope in the sense that $\hat{x}$ violates the inequalities. Separating inequalities can then be added to the relaxation to tighten

Let $S = \emptyset$.
Order $j_1, \ldots, j_n$ so that $\hat{x}_{j_1} \le \cdots \le \hat{x}_{j_n}$.
For $m = 1, \ldots, M$ while $S = \emptyset$:
    Let $J^m = \{j_1, \ldots, j_m\}$.
    Let $\bar{x}^1(J^m), \ldots, \bar{x}^N(J^m)$ be the undominated J^m-circuits generated
        by the greedy procedure of Fig. 1 with $J = J_+ = J^m$.
    For each $\{t_1, \ldots, t_m\} \subset \{1, \ldots, N\}$:
        Let $\sum_{i=1}^m a_{j_i} x_{j_i} = \alpha$ be an equation satisfied by $\bar{x}^{t_1}(J^m), \ldots, \bar{x}^{t_m}(J^m)$.
        If $\sum_{i=1}^m a_{j_i} \hat{x}_{j_i} < \alpha$ then add $\sum_{i=1}^m a_{j_i} x_{j_i} \ge \alpha$ to S.

Fig. 3. Separation heuristic for finding a set S of facet-defining inequalities with positive coefficients violated by a given point $\hat{x}$.

it.

Suppose first that we seek separating inequalities with all positive coefficients, so that $J = J_+$. Given a point $\hat{x}$ to be separated, let $j_1, \ldots, j_n$ be an ordering of variable indices for which $\hat{x}_{j_1} \le \cdots \le \hat{x}_{j_n}$. We consider the sequence of subsets $J^1, J^2, \ldots, J^n$ where $J^m = \{j_1, \ldots, j_m\}$. Beginning with J^1, we try to generate facet-defining inequalities corresponding to each J^m, until we find a separating inequality. For each J^m we use the greedy procedure of Fig. 1 to generate all undominated J^m-circuits with respect to $J^m = J^m_+$ and use these J^m-circuits to generate facet-defining inequalities as described earlier. Any of the resulting inequalities violated by $\hat{x}$ are separating. If none are separating, we move to J^{m+1} and repeat. The precise algorithm appears in Fig. 3. A similar algorithm is shown in [4] to be a complete separation procedure for the permutation polytope.

In practice, the algorithm would not continue all the way to J^n when no separating inequalities are found, because it is impractical to generate all facet-defining inequalities containing a large number of variables. Rather, the algorithm would stop at some predetermined maximum $m = M$, or when the number of number of undominated J^m-circuits becomes too large.

Let $S = J_+ = J_- = \emptyset$.
Order $j_1, \ldots, j_n$ so that
    $\min\{\hat{x}_{j_1} - v_1, v_n - \hat{x}_{j_1}\} \leq \cdots \leq \min\{\hat{x}_{j_n} - v_1, v_n - \hat{x}_{j_n}\}$.
For $j = 1, \ldots, M$:
    If $\hat{x}_{j_1} - v_1 \leq v_n - \hat{x}_{j_1}$ then add j to J_+.
    Else add j to J_-.
For $m = 1, \ldots, M$ while $S = \emptyset$:
    Let $J^m = \{j_1, \ldots, j_m\}$, $J_+^m = J^m \cap J_+$, $J_-^m = J^m \cap J_-$.
    Let $\bar{x}^1(J^m), \ldots, \bar{x}^N(J^m)$ be the undominated J^m-circuits generated
        by the greedy procedure of Fig. 1 with $J_+ = J_+^m$, $J_- = J_-^m$.
    For each $\{t_1, \ldots, t_m\} \subset \{1, \ldots, N\}$:
        Let $\sum_{i=1}^m a_{j_i} x_{j_i} = \alpha$ be an equation satisfied by $\bar{x}^{t_1}(J^m), \ldots, \bar{x}^{t_m}(J^m)$.
        If $\sum_{i=1}^m a_{j_i} \hat{x}_{j_i} < \alpha$ then add $\sum_{i=1}^m a_{j_i} x_{j_i} \geq \alpha$ to S.

Fig. 4. Separation heuristic for finding a set S of facet-defining inequalities with arbitrary coefficients violated by a given point $\hat{x}$.

As an illustration, suppose that $(\hat{x}_1, \ldots, \hat{x}_7) = (6, 2, 5.5, 7, 5.7, 8, 9)$ in example (17). This is not a feasible solution, if only because it does not consist of values from the domain $\{v_1, \ldots, v_7\} = \{2, 5, 6, 7, 9, 10, 12\}$. Here $(j_1, \ldots, j_7) = (2, 3, 5, 1, 4, 6, 7)$. For $J^1 = \{2\}$ we have the single facet-defining inequality $x_2 \geq 2$, but it does not separate $\hat{x}$. For $J^2 = \{2, 3\}$ we have undominated J-circuits (v_1, v_2) and (v_3, v_1), which define the facet-defining inequality $3x_2 + 4x_3 \geq 26$. It does not separate $\hat{x}$. But for $J^3 = \{2, 3, 5\}$ the greedy algorithm yields five undominated J-circuits:

$$
\begin{aligned}
\bar{x}^1(J^3) &= (v_1, v_2, v_3) \\
\bar{x}^2(J^3) &= (v_1, v_4, v_2) \\
\bar{x}^3(J^3) &= (v_3, v_1, v_2) \\
\bar{x}^4(J^3) &= (v_3, v_4, v_1) \\
\bar{x}^5(J^3) &= (v_4, v_2, v_1)
\end{aligned}
$$

Of the ten subsets of three J-circuits, three subsets (listed on the left below) define inequalities (listed on the right) that are satisfied by the other two undominated J-circuits:

$$\{\bar{x}^1(J^3), \bar{x}^2(J^3), \bar{x}^5(J^3)\}: \quad 8x_2 + 5x_3 + 10x_5 \geq 101$$

$$\{\bar{x}^1(J^3), \bar{x}^3(J^3), \bar{x}^5(J^3)\}: \quad 12x_2 + 11x_3 + 15x_5 \geq 169$$

$$\{\bar{x}^2(J^3), \bar{x}^4(J^3), \bar{x}^5(J^3)\}: \quad 6x_2 + 3x_3 + 8x_5 \geq 73$$

These are all the facet-defining inequalities containing three variables x_2, x_3, x_5 in positive terms. Because the first and third inequalities are violated by $\hat{x}$, they are separating cuts.

As noted earlier, the greedy heuristic requires running time $\mathcal{O}(m!mn \log n)$ for each J^m. If it generates N undominated J^m-circuits, we must solve $\binom{N}{m}$ systems of m equations. This is practical for small m.

The above heuristic can be modified slightly to generate separating inequalities with arbitrary signs. Rather than order the variables by nondecreasing size of $\hat{x}_j$, we can order them by nondecreasing size of $\min\{\hat{x}_j - v_1, v_n - \hat{x}_j\}$. Then we put $j \in J_+$ if $\hat{x}_j - v_1 \leq v_n - \hat{x}_j$ and $j \in J_-$ otherwise. The heuristic appears in Fig. 4.

11 Exploiting Cost Structure

One motivation for studying the circuit polytope for arbitrary domains is that it may allow us to exploit structure in a cost function that appears in the problem. A careful choice of the domain values can result in a tighter relaxation.

Suppose, for example, that the problem contains the cost function $\sum_i c_{ix_i}$ that appears in the traveling salesman problem (2). Associate each index i with a value v_i, and suppose that the costs c_{ij} have the property that, when the values v_i are properly chosen, $g(v_i, v_j) = c_{ij}$ is close to the value of an affine function $h(v_i, v_j)$ for $i < j$, and it is close to the value of an affine function $h'(v_i, v_j)$ when $j < i$. The v_is can be set to any nonnegative value, and the variables can be reordered if desired, to obtain a good affine fit. Then one can use computational geometry techniques to compute the convex hull of $S = \{(z, x_i, x_j) \mid z = g(x_i, x_j),\ x_i, x_j \in \{v_1, \ldots, v_n\}\}$. Consider all facets of the convex hull that are described by inequalities of the form

$$z \geq \beta_{0k} + \beta_{1k}x_i + \beta_{2k}x_j, \quad k \in K \tag{24}$$

Then all of the points of S are close to the facets described by (24).

Now let $Ax \geq b$ be a system of valid inequalities for the circuit polytope $C_n(v)$, where v is the vector of values just chosen. We can write a linear relaxation of the traveling salesman problem (2) that exploits the cost structure:

$$\begin{aligned} &\min \sum_{ij} z_{ij} \\ &z_{ij} \geq \beta_{0k} + \beta_{1k}x_i + \beta_{2k}x_j, \ \text{ for all } i, j \text{ and all } k \in K \\ &Ax \geq b \end{aligned} \tag{25}$$

For example, suppose the cost data c_{ij} are as in Table 3. If we let $(v_1, v_2, v_3) = (0, 1.5, 3.5)$, the values $g(v_i, v_j) = c_{ij}$ are close to the values of the affine function $h(v_i, v_j) = 4(v_i - v_j)$ for $i < j$ and close to $h'(v_i, v_j) = 4(v_j - v_i)$ for $j < i$. The convex hull of S has two facets of the form (24), namely

$$\begin{aligned} &z \geq \tfrac{26}{7}x_i - \tfrac{26}{7}x_j \\ &z \geq -\tfrac{26}{7}x_i + \tfrac{26}{7}x_j \end{aligned}$$

So if $Ax \geq b$ is a set of valid inequalities for $C_n(v)$, the relaxation (26) therefore becomes

Table 3
(a) Cost data c_{ij}. (b) Values of $h(x_i, x_j)$ when $x_i \le x_j$ and $h'(x_i, x_j)$ when $x_j \le x_i$.

(a)

	j			
		1	2	3
i	1	0	6	13
	2	6	0	9
	3	13	9	0

(b)

	x_j			
		0	1.5	3.5
x_i	0	0	6	14
	1.5	6	0	8
	3.5	14	8	0

$$\begin{aligned} &\min \sum_{i=1}^{3}\sum_{j=1}^{3} z_{ij} \\ &z_{ij} \ge \tfrac{26}{7}x_i - \tfrac{26}{7}x_j \ \text{ for all } i,j \in \{1,2,3\} \\ &z_{ij} \ge -\tfrac{26}{7}x_i + \tfrac{26}{7}x_j \ \text{ for all } i,j \in \{1,2,3\} \\ &Ax \ge b \end{aligned} \tag{26}$$

If c_{ij} is a distance, it may be possible to exploit the structure of the distance metric, particularly if it is rectilinear. Further details, along with an application to the quadratic assignment problem, may be found in [4].

12 Conclusions and Future Research

We showed that there is a strong relationship between the facets of the circuit polytope and undominated partial circuits. Specifically, the facet-defining inequalities containing a given set J of at most $n-4$ terms are precisely the valid inequalities defined by sets of undominated J-circuits. Furthermore, the undominated J-circuits can be generated with a greedy procedure.

This means that all facet-defining inequalities with terms in J can in principle be found by a two-phase procedure. A combinatorial phase uses the greedy algorithm to generate all undominated J-circuits. A numerical phase then computes inequalities that are satisfied at equality by affinely independent subsets of the undominated J-circuits. It checks them for validity by checking whether the other undominated J-circuits satisfy them. The first phase is independent of the domain values $v_1, \ldots, v_n$, but the second is not. This two-phase procedure can be viewed as isolating the discrete and continuous aspects of the circuit polytope.

This leads to a fast separation algorithm for sparse facets. The combinatorial phase uses the greedy algorithm to generate all undominated J-circuits for heuristically chosen small sets J. The numerical phase computes the corresponding facet-defining inequalities and checks if any are separating.

We also identified a family of permutation facets that are circuit facets and explicitly described all two-term circuit facets. We showed how the circuit constraint with arbitrary variable domains can exploit cost structure in the objective function.

These results presented here lay the theoretical groundwork for the solution of sequencing problems with the help of linear relaxations comprised of circuit inequalities. Computational testing is the next step, together with investigation of how the separation heuristics can be tuned or altered to achieve best results. The cost matrices of typical problems can be examined to determine the extent to which cost can be approximated as an affine function or a rectilinear metric, to allow an effective choice of domain values.

An interesting research question is whether circuit inequalities can be profitably converted to 0-1 inequalities and combined with known traveling salesman inequalities. For a given domain $\{v_1, \ldots, v_n\}$, the conversion could be based on the identity $x_i = \sum_j v_j y_{ij}$, where y_{ij} is the 0-1 variable that appears in the traveling salesman model (3).

Our primary goal, however, has been to explore the structure of the circuit polytope in the original space, as an alternative to the conventional 0-1 representation.

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