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The Effect of Refinancing Costs and Market Imperfections on the Optimal Call Strategy and the Pricing of Debt Contracts

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This article, which was originally written in 1986, develops a methodology for valuing mortgage-backed securities with refinancing costs. We solve simultaneously for the valuation of the mortgage-backed security (loan) and the borrower's refinancing strategy, pricing all coupon levels simultaneously. Because the borrower may refinance his or her loan and incur costs at many times in the future, the optimal refinancing decisions arise from an optimal dynamic strategy that reflects the costs of all potential future refinancings. Though the borrower faces multiple rounds of refinancing costs, the market value of the loan cannot exceed the call price plus a single round of refinancing costs.

An interesting feature of the market prices of mortgage-backed securities and corporate bonds has been the high premiums above par value observed for high coupon securities. For example, the substantial evidence of large premiums in the mortgage market (see Figure 1) as well as direct observation of refinancing costs (*e.g.*, attorney, document and title fees as well as points paid in advance to the servicer) suggests that refinancing costs are themselves quite large, reducing the incentive for many mortgage borrowers to refinance their loans. We model the effect of refinancing costs on the optimal refinancing decision in order to examine the determinants of these high premiums and the relative pricing of securities with different coupon rates. The borrower's liability, which determines his or her refinancing strategy, is valued simultaneously with the marketed loan. We derive an explicit bound on the pricing of debt contracts and demonstrate that plausible values for refinancing costs lead to violations of this bound and are too small to fully explain the observed premiums within our model.[1]

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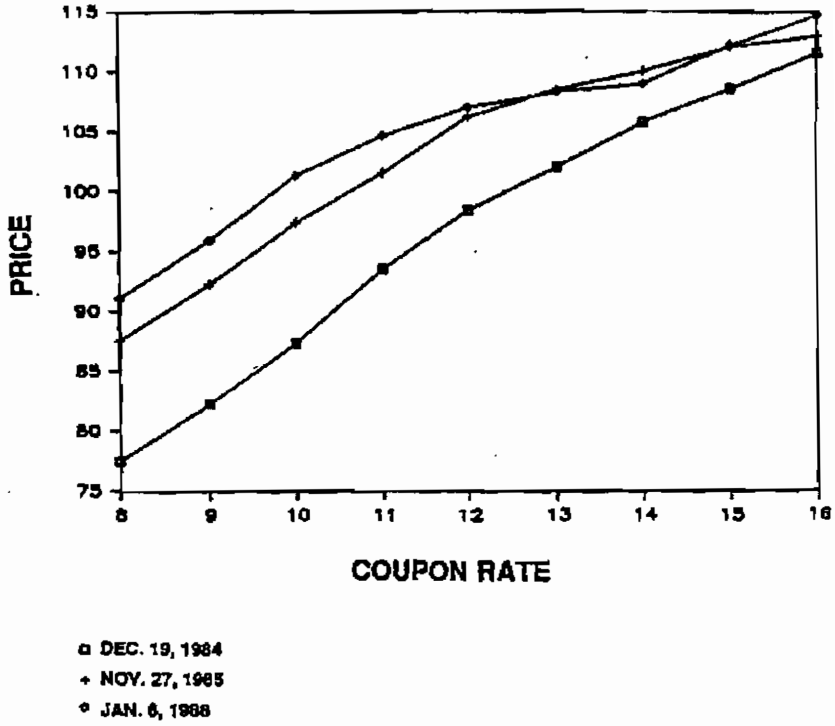
[1] In an insightful empirical analysis Stanton (1995) uses generalized method of moments to estimate the distribution of refinancing costs. His results also suggest that simple refinancing cost models lead to estimated refinancing costs that are implausibly high.

Figure 1 ■ GNMA prices as a function of coupon rates.



In the presence of refinancing costs, a borrower's optimal refinancing strategy reflects both the immediate refinancing costs incurred upon refinancing as well as the future benefits and costs expected under a dynamic optimal refinancing policy. This explicitly takes into account the possibility that additional refinancing can be optimal in the future.[2] A decision to refinance lowers the borrower's coupon payment, thereby reducing the borrower's incentive to pay the refinancing costs required to refinance again in the near future (*e.g.*, lengthening the distribution of the first subsequent refinancing time).[3] With refinancing costs, the value of the borrower's optimal strategy is influenced by the possibility that rates will be lower in the near future and that by waiting to repay the existing loan it may be possible to obtain a lower interest rate than that which is available if

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[2] Kraus (1973) assumes that the future loans do not have transaction costs to argue that the optimal current refinancing decision with transaction costs is independent of the future financing.

[3] At an optimal refinancing point a borrower typically will place value on the option to later call the new financing. As a consequence of the declining interest rates since the early 1980s, many fixed-rate borrowers have repeatedly refinanced.

the existing loan is refinanced now.[4] Because a current decision to refinance reduces the potential future gains from subsequent refinancing, refinancing costs and call premia cause borrowers to become "locked" into contracts. In terms of the borrower's refinancing strategy, the immediate gain from refinancing must equal the refinancing costs and call premium at the optimal refinancing point. Therefore, at this point the value of the borrower's refinancing strategy equals the sum of the refinancing costs (paid to third parties), call premium and the value of the refinancing strategy at the refinanced coupon. There is a "magnification" of the refinancing costs paid to third parties in terms of the borrower's refinancing strategy because the value of the strategy exceeds the call price plus refinancing costs if there is a positive probability of later refinancing costs. The potential total costs exceed those from a single refinancing, because we include future as well as current refinancing decisions in our analysis. In terms of the value of the marketed loan, however, there is no magnification of the costs. In particular, we establish that the market price must be less than the sum of the cost of a single refinancing plus the call price, even when multiple future refinancings are possible.[5] This contrasts with the interpretation that the boundary can be implemented by simply adding transaction costs to the call price to modify the zero-transaction-cost boundary condition that the value of the loan does not exceed the call price.[6]

Because the optimal current refinancing decision depends upon the optimal future refinancing decisions on subsequent debt contracts (if callable), the value of a debt contract depends upon its future values and the future values of contracts that could arise from any possible subsequent refinancing decisions (*i.e.*, only those contracts with lower coupons). All of these instruments must be valued simultaneously, though recursively and in the order of increasing coupon payment. We study the call strategies and valuation of both nonamortizing coupon bonds (such as corporate bonds) and self-amortizing instruments (such as mortgage-backed securities).

Due to the refinancing costs, debt contracts can sell at a price above the call price.[7] But at a point at which refinancing is optimal the market value of the debt contract is the call price (*e.g.*, often par) because the debt owner receives the call price at this point. Therefore, as illustrated in Figure 2, in the presence

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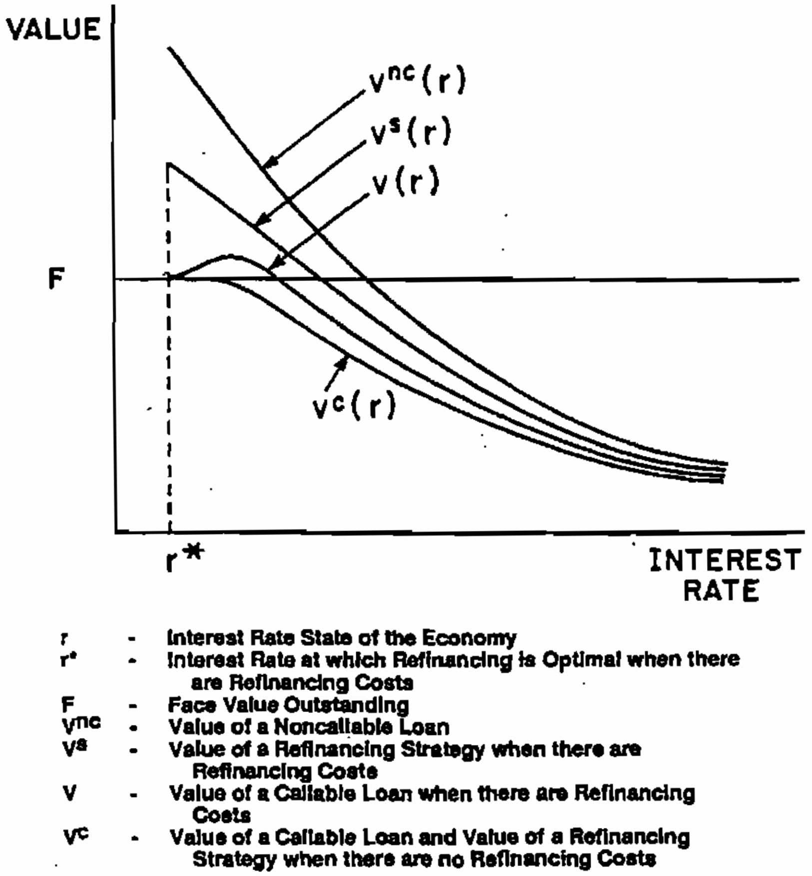
[4] Hendershott, Hu and Villani (1983) suggest in a footnote that it can be optimal for a borrower to postpone refinancing if there are positive transaction costs though immediate refinancing would be optimal absent these costs. However, they do not model this issue.

[5] The analysis in Stanton (1995), which builds from our theoretical analysis, permits only a single refinancing opportunity and does not develop this bound.

[6] The bound we derive shows that such a boundary condition would not be reached.

[7] Dunn and Spatt (1999) show that many of the qualitative features of debt pricing will be robust to the presence of refinancing costs.

Figure 2 ■ Value as a function of the interest rate state.



of refinancing costs the market value of the debt contract can actually decline locally with decreases in interest rates for high coupon rate instruments (this observation was originally noted by Kraus (1983)).[8] As illustrated by Figure 1, in the mid-1980s the prices of 14% GNMA (Government National Mortgage Association) pass-through securities declined with a decrease in interest rates.[9]

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[8] In an analysis of optimal tax realization behavior, Dammon and Spatt (1996) show that the introduction of transactions costs alters many of their model's comparative statics properties.

[9] For example, in early 1986 some Wall Street traders hedged long positions in high coupon mortgage-backed securities by shorting Treasury bonds. Because the correct hedge would have required a long position in the Treasuries, these firms lost money on both sides of their actual position as interest rates declined.

Even if the possibility that the value of the high coupon instrument decreases with declines in the interest rate is not realized, the value of the high coupon instrument will be less sensitive to the movements in the state variables than lower coupon instruments.

The optimal refinancing decision rule as a function of the coupon payment on the existing loan is monotone in the coupon payment. It is optimal to refinance if and only if the coupon payment is at least a specified level because the gain from refinancing increases in the existing coupon payment and the cost does not depend on the existing payment. However, the market value of the debt contract can actually decrease with an increase in the coupon payment because when the coupon is sufficiently high that refinancing is optimal, the price of the debt contract is only the call price. If the coupon payment is somewhat lower so that refinancing is not optimal at the current market rate, then the contract can be worth more than the call price.

Our analysis builds upon contingent claim valuation analysis in the form of equilibrium term structure modeling (see Cox, Ingersoll and Ross 1985), the application of equilibrium term structure models to the valuation of mortgage-backed securities (*e.g.*, Dunn and McConnell 1981a,b) and the analysis of optimal call strategies without refinancing costs (see Brennan and Schwartz 1977, Ingersoll 1977a). Ingersoll (1977b) uses a contingent claim valuation approach in the presence of underwriting costs to examine callable convertible instruments, and Timmis (1985) and Stanton (1995) also examine the effect of refinancing costs on the valuation of mortgage-backed securities, but they do not consider the possibility of additional subsequent refinancing. That the optimal current refinancing decision in the presence of refinancing costs is influenced by its effect upon future refinancing possibilities and that the optimal current decision reflects such nonmyopic effects has been emphasized in the literature (*e.g.*, Weingarten 1967, Elton and Gruber 1971, 1972, 1975, Kalyman 1971). In fact, Elton and Gruber (1975) observe that it is optimal for the issuer to refund the debt if and only if the current market interest rate does not exceed a specified critical level, and Kalyman (1971) discusses the dependence of the refinancing decision upon the new coupon rate available in the market. While the earlier dynamic programming analyses of bond refunding with refinancing costs solved for the optimal refinancing strategy taking prices as fixed, by incorporating contingent claims valuation considerations we can solve simultaneously for the market value of the debt contract and the optimal refinancing strategy and obtain general qualitative restrictions. The presence of refinancing costs changes the qualitative nature of the valuation of the marketed security.

The theoretical analysis of the valuation of fixed-income securities with refinancing costs is developed and the qualitative features of the solution are explored in the following section. The numerical solution algorithm is briefly described in the third section. The fourth section examines the degree to which refinancing costs can explain the magnitude of premiums on high coupon mortgage-backed securities. We offer concluding comments in the final section.

Theoretical Analysis

Consider a borrower with a loan that has a coupon payment denoted by c, interest rate state vector of $\mathbf{r}$ and remaining term to maturity of t. The value of the borrower's refinancing strategy, $V^s(\mathbf{r}, t; c)$, is the value of the interest and principal payments and call premiums on the existing loan as well as the refinancing costs of switching to lower coupon rate contracts from the current and any future loans during the period of time between now and the current loan's maturity. The difference between $V^s(\mathbf{r}, t; c)$ and the market value of the existing loan, $V(\mathbf{r}, t; c)$, is the present value of the refinancing costs (and any loan servicing fees) which are paid by the borrower to third parties and, therefore, are not part of the cash flows to investors. This present value of the refinancing costs will exceed the present value cost of a single refinancing if there is a positive probability of multiple refinancing. $V^s(\mathbf{r}, t; c)$ is the value of a synthetic security representing the value of a modified fixed-rate loan (or an adjustable-rate loan with the coupon payment capped at c) that allows the borrower the option to adjust the coupon payment downward to the then-prevailing market coupon payment by paying a fee equal to the refinancing cost.

We define the face or par value of the debt as a predetermined function of time, that is, $F(t; c)$, where t is the time remaining until maturity and c is the contractual coupon payment; we let $P(t)$ denote the call premium ($P(t) \geq 0$), restricting the call premium to be a function of the time remaining to maturity and not dependent upon the time since the most recent refinancing or the coupon payment on the existing contract. The market coupon payment (*e.g.*, for a refinanced security) is $cm(\mathbf{r}, t)$ at time t in state $\mathbf{r}$. By definition, the market coupon payment $cm(\mathbf{r}, t)$ is the lowest payment such that the marketed debt contract is worth par in interest state $\mathbf{r}$ and remaining term to maturity t, that is, $V(\mathbf{r}\ t; cm(\mathbf{r}, t)) = F(t; c)$. Finally, the refinancing cost at time t in state $\mathbf{r}$ is denoted by $T(\mathbf{r}, t)$.

We make the following assumptions.

(A1) Capital markets are frictionless with no taxes, information costs or transactions costs for purchasing or selling securities. Refinancing costs and any loan service fees are paid to third parties outside of the capital market.

(A2) A vector of state variables, $\mathbf{r}$, completely describes the state of the economy. It is not required that the sample path of the state variable(s) be continuous.

(A3) Borrowers are homogeneous and face identical distributions of gains from moving. These gains from moving result in a "forced" prepayment if the gain exceeds the costs of an early prepayment, where the costs depend upon the borrower's coupon.

Many of the results do not require Assumption (A3), but by assuming that there is a single mortgage contract offered in the market a borrower's private information does not influence the selection of the mortgage contract (see Dunn and Spatt 1988). One of the notable characteristics of mortgage borrowers is that, in practice, many of them call their loans even when the market interest rate is above the contract rate on their existing loans. Though such calls are not optimal with respect to financing costs, these prepayments (which generally occur when a borrower changes his residence and the existing mortgage loan is not assumed by the purchaser of the house) can form a constrained maximum from the perspective of a rational borrower.[10] For example, in the context of mortgage loans with a due-on-sale provision, the borrower's utility gain from transferring his debt obligation would motivate prepayment by borrowers when the market interest rate exceeds the contract rate, as in Dunn and Spatt (1985).

For some purposes Assumption (A3) is replaced with Assumption (A3′):

(A3′) Borrowers are homogeneous and never have forced prepayments. The sole reason for refinancing is to reduce interest payments.

(A4) The loan contracts do not have any default risk or service fees.

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[10] In an empirical study of the calls of nonconvertible corporate bonds Vu (1986) argues that calls are relatively more common before, rather than after, the value of the loan reaches par value. The calls that occur when the value of the debt is less than par value are analogous to voluntary prepayments by mortgage borrowers that occur despite a low loan rate. In Vu's (1986) sample, the early calls are often motivated by the desire of equityholders to eliminate restrictive covenants. These calls are somewhat similar to the calls by mortgage borrowers that are motivated by their desire to transfer the mortgaged property. However, Vu's sample cannot be used to discern the relative importance of calls motivated by a desire to refinance at a lower interest rate and calls motivated by a desire to eliminate restrictive covenants because in the 1962–1977 period he considers there were not large declines in interest rates. In fact, the choice of sample period is critical to judging the relative empirical importance of calling to obtain a lower interest rate and calling to eliminate restrictive covenants such as restrictions on the right to transfer the mortgaged asset. We anticipate a very different conclusion over the last 20 years.

This eliminates the possibility of a borrower not refinancing despite a substantial decline in interest rates solely because the borrower cannot qualify for a new loan at those terms. The assumption of no service fees simplifies the exposition.

We note that our model is not a full general equilibrium specification of the valuation of debt contracts. In particular, though we value the marketed loans and refinancing strategy (a synthetic security) assuming that there are no arbitrage opportunities in the capital market, our model does not explain why borrowers use contracts with substantial expected refinancing costs.[11]

Dunn and McConnell (1981a,b) derive the partial differential equation describing the evolution of the market value of a mortgage loan in which forced prepayment of the loan is modeled as a Poisson process, the instantaneous interest rate follows a mean-reverting square root process (as in the one-factor model developed by Cox, Ingersoll and Ross (1985)) and there are no refinancing costs. Refinancing costs would affect the boundary condition, but these costs do not alter the partial differential equation describing the value of the loan or the underlying interest rate dynamics. Under our assumption of no service fees, the partial differential equations for the value of the marketed loan and the refinancing strategy would be identical. Because the central features in modeling costly refinancing are the refinancing boundary, the interaction between the value of the refinancing strategy and the marketed loan as well as the determination of the refinancing rate, we focus our discussion on these qualitative aspects of the pricing function.

To value both the debt instrument and the refinancing strategy one solves backward from the terminal boundary (the maturity of the bond) and the upper and lower boundaries in the state space. At the terminal boundary both the refinancing strategy and marketed claim are worth the terminal balance on the loan, that is, par for a coupon bond and zero for a self-amortizing mortgage-backed security. Therefore,

$$V(\mathbf{r}, 0; c) = F(0; c) \tag{1}$$

$$V^s(\mathbf{r}, 0; c) = F(0; c). \tag{2}$$

In contrast to mortgage loans, conventional corporate coupon bonds are typically nonamortizing. The balloon payment at the maturity of the bonds yields a slightly different terminal boundary condition than for self-amortizing mortgages. While corporate indebtedness is typically risky (unlike the GNMAs,

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[11] Stanton and Wallace (1998) observe that lower interest rate, higher points contracts have lower refinancing costs (due to the less valuable refinancing option) than higher interest rate, lower points contracts.

which are subject to the "full faith and credit" of the United States), we abstract from default risk.

The objective of the borrower is to adopt the refinancing strategy that minimizes the present value of the synthctic security describing the borrower's liability, $V^s(\mathbf{r}, t; c)$. The borrower will refinance when the benefit equals or exceeds the total cost of refinancing. The benefit of refinancing equals the difference between the value of the refinancing strategy at the existing contract's coupon payment and the value of the refinancing strategy with the current market coupon payment, that is,

$$BENEFIT = V^s(\mathbf{r}, t; c) - V^s(\mathbf{r}, t; cm(\mathbf{r}, t)).$$

The total cost of refinancing is the present value of the total refinancing costs paid to third parties, $T(\mathbf{r}, t)$, plus any call premium, $P(t)$, paid to the owners of the current loan contract, that is,

$$COST = T(\mathbf{r}, t) + P(t).$$

Because it would be optimal to immediately refinance if $BENEFIT > COST$, the optimal refinancing strategy guarantees that $BENEFIT \leq COST$, which implies

$$V^s(\mathbf{r}, t; c) \leq V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) + T(\mathbf{r}, t) + P(t). \tag{3}$$

Refinancing occurs when the value of the refinancing strategy at an existing coupon payment equals the value of the strategy at the current market coupon payment plus the refinancing cost and call premium.[12] At this refinancing point, the value of the underlying loan equals its call price, that is,

$$V(\mathbf{r}^*, t; c) = F(t; c) + P(t), \tag{4}$$

where $\mathbf{r}^*$ is the value of $\mathbf{r}$ such that

$$V^s(\mathbf{r}^*, t; c) = V^s(\mathbf{r}^*, t; cm(\mathbf{r}^*, t)) + T(\mathbf{r}^*, t) + P(t), \tag{5}$$

and $P(t) = 0$ for GNMAs as their underlying loans do not have a prepayment penalty. The dynamics of the value of the refinancing strategy and the effect of the refinancing boundary on the value of the underlying debt contract is illustrated in Figure 3.

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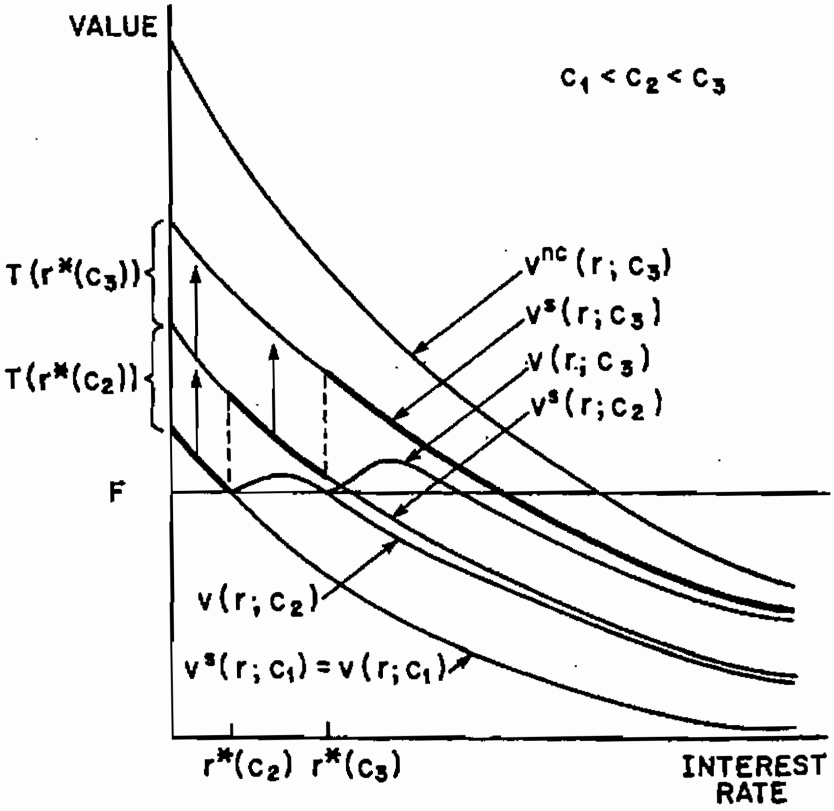
[12] If refinancing costs are zero on the current and future loans, then $V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) = V(\mathbf{r}, t; cm(\mathbf{r}, t))$ and the inequality in (3) simplifies to $V(\mathbf{r}, t; c) \leq F(t; c) + P(t)$. Hence, the market value of the loan does not exceed its call price, which is the standard call boundary without refinancing costs.

Figure 3 ■ Illustration of refinancing behavior.

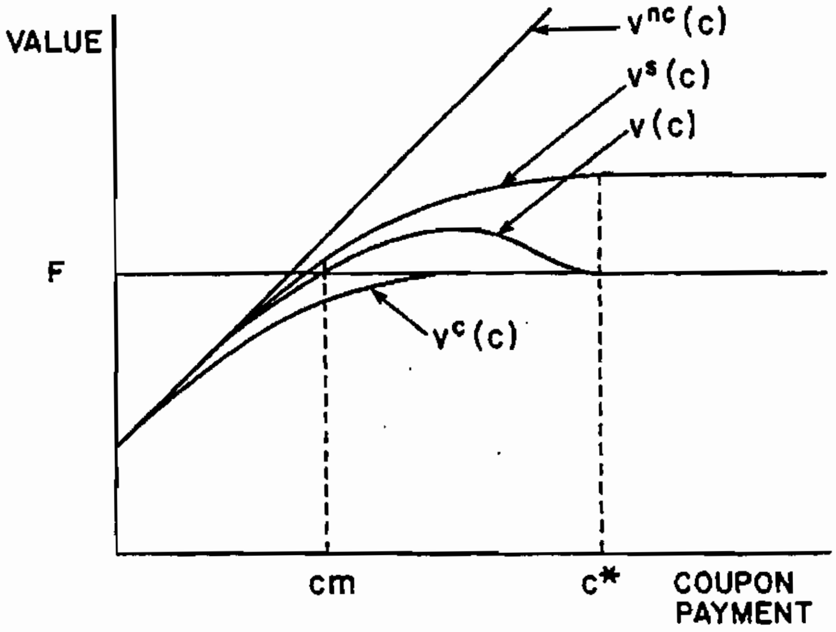


The form of the refinancing boundary yields implications on the effect of the coupon payment on the value of the refinancing strategy and the underlying loan. Below we examine a number of features of the refinancing strategy which are illustrated in Figure 4.

Proposition 1: For values of c for which it is not optimal to refinance, $V^s(\mathbf{r}, t; c)$ is strictly increasing in c, that is, $V^s(\mathbf{r}, t; c_h) > V^s(\mathbf{r}, t; c_l)$ for $c_h > c_l$.

Proof: Consider two loans with the same face values and with coupon payments c_h and c_l $(c_h > c_l)$ for which refinancing is not optimal. Construct a value function, $\hat{V}^s(\mathbf{r}, t; c_l)$, for the refinancing strategy of the low coupon payment borrower using the optimal intertemporal refinancing strategy of the high coupon borrower, including any forced prepayments, that is, refinancing due to gains from moving. The V^s and $\hat{V}^s$ functions are affected by prepayment

Figure 4 ■ Value as a function of coupon payment.



F	-	Face Value Outstanding
c	-	Coupon Payment per F Dollars of Face Value
cm	-	Market Coupon Payment
c*	-	Minimum Coupon Payment for Optimal Refinancing
V^{nc}	-	Value of a Noncallable Loan
V^{s}	-	Value of a Refinancing Strategy when there are Refinancing Costs
V	-	Value of a Callable Loan when there are Refinancing Costs
V^{c}	-	Value of a Callable Loan and Value of a Refinancing Strategy when there are no Refinancing Costs

decisions due to moving, but do not include the utility gain of the borrower due to moving. These functions represent the market value of synthetic securities. Because the refinancing decisions of the low coupon borrower are constrained to be the same as those optimally selected by the high coupon borrower, both borrowers face the same future cash flows after the initial refinancing. However, until the first refinancing point is reached, the high coupon borrower incurs more costs because he pays higher coupons and a higher interest rate on his balance. Therefore, $V^s(\mathbf{r},t;c_h) > \hat{V}^s(\mathbf{r},t;c_l)$. From the definition of $V^s(\mathbf{r}, t; c_l)$ as the value of the optimal refinancing strategy of the low coupon borrower, it follows that $\hat{V}^s(\mathbf{r},t;c_l) \geq V^s(\mathbf{r}, t; c_l)$, which implies $V^s(\mathbf{r}, t; c_h) > V^s(\mathbf{r}, t; c_l)$. Q.E.D.

Proposition 2: Refinancing is optimal and $V^s(\mathbf{r}, t; c) = V^s(\mathbf{r}, t; c^*)$ for coupons $c \geq c^*$, where c^* is the minimum coupon satisfying $V^s(\mathbf{r}, t; c^*) = V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) + T(\mathbf{r}, t) + P(t)$.

Proof: By Proposition 1, if it is not optimal for the borrower to refinance immediately, the value of the borrower's refinancing strategy strictly increases in the existing coupon payment, c. If the borrower refinances immediately, then the current market coupon payment, $cm(\mathbf{r}, t)$, and the value of the refinancing strategy just prior to refinancing, which equals $V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) + T(\mathbf{r}, t) + P(t)$, are independent of the existing coupon. Therefore, a higher current coupon increases the net benefit of refinancing, which equals zero at c^*. Because the objective of the borrower is to minimize the value of the refinancing strategy, it is optimal to refinance if and only if $c \geq c^*$. In the refinancing region (*i.e.*, $c \geq c^*$) $V^s(\mathbf{r}, t; c) = V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) + T(\mathbf{r}, t) + P(t)$. Q.E.D.

Only if the sample path of the state variables is discontinuous can the borrower reach a c for which $c > c^*$.

Corollary 1: As the coupon on the existing contract rises, refinancing is optimal in at least as many (*i.e.*, not fewer) states of the world for a given borrower.

Proof: By Proposition 2, if prepayment is optimal at $(\mathbf{r}, t)$ for coupon c_2, then it is also optimal for any $c_1 > c_2$. Q.E.D.

These descriptions of the impact of the coupon on the value of the refinancing strategy do not depend upon the stochastic process generating the movement of interest rates or the form of $T(\mathbf{r}, t)$. For example, the refinancing costs can depend upon the state variables, or the refinancing costs can be identically zero. In addition, the assumption that the call price is not influenced by the coupon payment can be relaxed by assuming that the call price is nondecreasing in the coupon (for corporate bonds the call price schedule sometimes increases in the coupon).

Corollary 2: For any $c \geq c^*$, the market value of the loan equals its call price (face value when $P(t) = 0$), that is, $V(\mathbf{r}, t; c) = F(t; c) + P(t)$ for $c \geq c^*$.

Proof: This follows immediately from Proposition 2 and the value of the loan equaling its call price at the refinancing boundary. Q.E.D.

Proposition 3: For values of c satisfying $cm(\mathbf{r}, t) < c < c^*$, the market value of the loan exceeds its face value, that is, $V(\mathbf{r}, t; c) > F(t; c)$.

Proof: The market value of the cash flows absent immediate refinancing must be larger for c than $cm(\mathbf{r}, t)$ because $c > cm(\mathbf{r}, t)$. At the refinancing boundary, the value of the debt satisfies $V(\mathbf{r}, t; c) = F(t; c) + P(t) \geq F(t; c)$. While a higher coupon bond implies that the debt is closer to reaching the refinancing boundary, the immediate coupon benefit ensures that the value of the loan strictly exceeds par if refinancing is not optimal. Q.E.D.

Of course, if transactions costs are identically zero, and if there is no call premium, then condition (3) and the monotonicity of V^s in c implies that refinancing is optimal if and only if $c \geq cm(\mathbf{r}, t)$, that is, $c^* = cm(\mathbf{r}, t)$. The monotonicity of V^s in c together with the characterization for refinancing to be optimal implies that with a positive call premium or refinancing cost, then $c^* > cm(\mathbf{r}, t)$ so that the reduction in coupon is large enough to compensate for the refinancing costs and call premia.

Proposition 3 implies that the value of a loan without a call premium (*i.e.*, $P(t) = 0$, as is the case for mortgage-backed securities) *locally decreases* in the coupon payment as the refinancing boundary is approached. For example, the restriction $P(t) = 0$ implies $V(\mathbf{r}, t; c^*) = F(t; c)$, while $V(\mathbf{r}, t; c) > F(t; c)$ holds for a somewhat smaller c. A similar result holds for $P(t) > 0$.

Proposition 4: The value of the refinancing strategy can exceed the call price plus the current refinancing costs. This "magnification" effect is caused by the possibility of future refinancing costs.

Proof: The value of the refinancing strategy must be at least as large as the market value of the loan because the cash flows are at least as large. The only difference between the cash flows are that the refinancing strategy includes the refinancing costs on the current and future loans. Therefore, $V^s(\mathbf{r}, t; c) \geq V(\mathbf{r}, t; c)$. By construction of $cm(\mathbf{r}, t)$ (hereafter we suppress the arguments of cm), immediately after refinancing $V(\mathbf{r}, t; cm) = F(t; c)$. Therefore, $V^s(\mathbf{r}, t; c) \geq V(\mathbf{r}, t; c)$ implies that immediately after refinancing $V^s(\mathbf{r}, t; cm) \geq F(t; c)$. The conditions $V^s(\mathbf{r}, t; c) \geq V(\mathbf{r}, t; c)$ and $V^s(\mathbf{r}, t; c) \geq F(t; c)$ immediately after refinancing are satisfied with equality if and only if the future expected refinancing costs are zero. When $V^s(\mathbf{r}, t; cm) > F(t; c)$, it follows that $V^s(\mathbf{r}, t; c^*)$ is greater than $F(t; c) + T(\mathbf{r}, t) + P(t)$, so that the possibility of future refinancing delays the optimal refinancing point. Q.E.D.

The future expected refinancing costs can be zero if and only if either the refinancing charge is zero or refinancing is not optimal in the future no matter how low the future interest rate state (*e.g.*, at the terminal boundary $V^s(\mathbf{r}, 0; c) = V(\mathbf{r}, 0; c) = F(0; c)$).

We now establish the concavity of V^s in the coupon payment by restricting borrowers on the extreme coupons to the optimal exercise policy on an intermediate coupon. The restricted value is linear in c and the use of optimal refinancing behavior on the extreme coupons (reducing V^s) strengthens the concavity. For the remainder of this section we replace (A3) with (A3′).

Proposition 5: $V^s(\mathbf{r}, t; c)$ is concave in c, that is, $V^s(\mathbf{r}, t; c_2) \geq \alpha V^s(\mathbf{r}, t; c_1) + (1 - \alpha)\, V^s(\mathbf{r}, t; c_3)$, where $c_2 = \alpha c_1 + (1 - \alpha)c_3$ for $0 < \alpha < 1$ and $c_1 < c_2 < c_3$.

Proof: We compare the payments from a loan contract with coupon c_2 and a portfolio of loans with coupons c_1 and c_3, where $c_2 = \alpha c_1 + (1 - \alpha)c_3$ for $0 < \alpha < 1$ and $c_1 < c_2 < c_3$. The payments from the c_2 loan and the portfolio of c_1 and c_3 loans are equal until the first time when one of the loans is refinanced. Suppose that borrowers with coupons c_1 and c_3 use the c_2 refinancing strategy, so that all three loans are refinanced at the same time and refinanced into identical contracts. We let $\hat{V}^s(\mathbf{r}, t; c_j)$, denote the value of the restricted refinancing strategy of a borrower with coupon c_j for $j = 1, 3$. Then all future cash flows from the two portfolios are identical so that

$$V^s(\mathbf{r}, t; c_2) = \alpha \hat{V}^s(\mathbf{r}, t; c_1) + (1 - \alpha)\hat{V}^s(\mathbf{r}, t; c_3).$$

However, by switching from the c_2 strategy to their optimal strategies, the borrowers with coupon payments c_1 and c_3 cannot be worse off and may be able to reduce V^s. That is, $V^s(\mathbf{r}, t; c_j) \leq \hat{V}^s(\mathbf{r}, t; c_j)$ for $j = 1, 3$, and, therefore $V^s(\mathbf{r}, t; c_2) \geq \alpha V^s(\mathbf{r}, t; c_1) + (1 - \alpha)V^s(\mathbf{r}, t; c_3)$. Q.E.D.

The difference between the value of the refinancing strategy and the value of the marketed loan represents the value of the future refinancing costs (any loan service fees would also be included in the difference, if they were present). The current value of these refinancing costs increases as the borrower's coupon gets closer to the coupon at which refinancing is optimal.

Proposition 6: $V^s(\mathbf{r}, t; c) - V(\mathbf{r}, t; c)$ is weakly increasing in c, that is, $V^s(\mathbf{r}, t; c_h) - V(\mathbf{r}, t; c_h) \geq V^s(\mathbf{r}, t; c_l) - V(\mathbf{r}, t; c_l)$ for all $c_h > c_l$.

Proof: Throughout the proof we fix the state and time and therefore suppress the dependence of the value functions upon $(\mathbf{r}, t)$. $V^T(c)$ is the present value of the refinancing costs from following the optimal refinancing strategy for a loan with payment c, so that $V^T(c) = V^s(c) - V(c)$. To prove the property by contradiction, suppose that $V^T(c_l) > V^T(c_h)$, where $c_l < c_h$. Then the borrower with the c_l contract could switch to the strategy for the c_h contract and be better off (*i.e.*, have a lower V^s), contradicting the optimality of the c_l strategy if

$V^T(c_l) > V^T(c_h)$. Let $cm(c_h)$ denote the market coupon payment when c_h is refinanced. At this refinancing point $V^s(c_h) = F + P(t) + V^T(c_h)$ because $V(cm) = F$. If $c_l > cm(c_h)$, then $V(c_l) > F$ by Proposition 3. But, if $V^T(c_l) > V^T(c_h)$, then $V^s(c_l) = V(c_l) + V^T(c_l) > V^s(c_h)$, which contradicts $V^s(c_l) \leq V^s(c_h)$ by Propositions 1 and 2. If $c_l \leq cm(c_h)$, then $V(c_l) \leq F$ when c_h refinances. In this case c_l can save the current refinancing cost of the c_h strategy and follow the c_h strategy thereafter and have both a lower V^s and a lower V^T. Hence, $V^T(c_l) \leq V^T(c_h)$ for all $c_l < c_h$. Q.E.D.

The result in Proposition 6 can be used to derive an upper bound on the market value of the loan contract. Despite the possibility that the borrower refinances on several additional occasions, the value of the loan cannot exceed face plus a single refinancing cost. In contrast to the value of the refinancing strategy, there is no "magnification" of the transactions costs with respect to the market value of the loan.

Proposition 7: The market value of a loan cannot exceed its call price plus the current refinancing cost, that is, $V(\mathbf{r}, t; c) \leq F(t; c) + P(t) + T(\mathbf{r}, t)$, and the inequality is strict if $T(\mathbf{r}, t) > 0$.

Proof: We let c^* denote the coupon payment at which refinancing is optimal. The market offers a new coupon payment of $cm(\mathbf{r}, t)$, where $cm(\mathbf{r}, t) < c^*$ is implied by $T(\mathbf{r}, t) > 0$. We let c_h define the coupon value that maximizes $V(\mathbf{r}, t; c)$. The characterization of the optimal refinancing boundary implies $V^s(\mathbf{r}, t; c^*) = V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) + T(\mathbf{r}, t) + P(t)$.

Because $c_h \leq c^*$, Proposition 1 (the monotonicity of V^s in the coupon rate for $c \leq c^*$) implies

$$V^s(\mathbf{r}, t; c_h) \leq V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) + T(\mathbf{r}, t) + P(t). \tag{*}$$

By Proposition 6 (the monotonicity of $V^T(\mathbf{r}, t; c)$ in c) we have

$$V^s(\mathbf{r}, t; c_h) - V(\mathbf{r}, t; c_h) \geq V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) - V(\mathbf{r}, t; cm(\mathbf{r}, t))$$

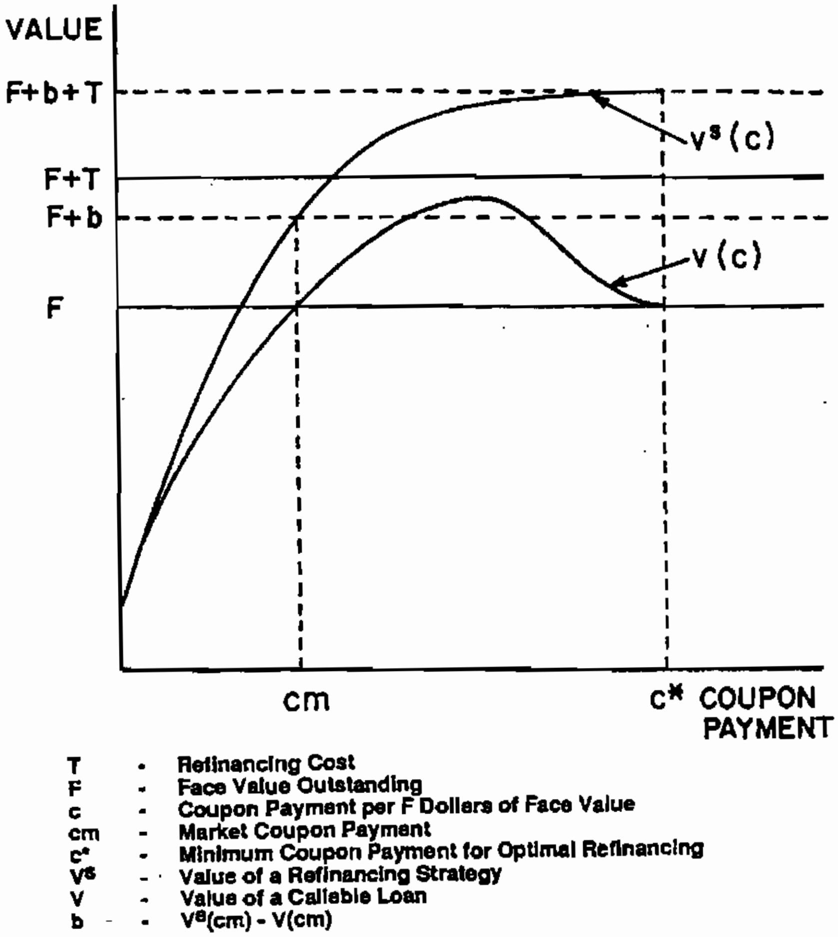
or

$$V(\mathbf{r}, t; c_h) \leq V^s(\mathbf{r}, t; c_h) - [V^s(\mathbf{r}, t; cm(\mathbf{r}, t)) - F]. \tag{**}$$

Combining (*) and (**) implies $V(\mathbf{r}, t; c_h) \leq F + T(\mathbf{r}, t) + P(t)$.

By the definition of c_h, $V(\mathbf{r}, t; c) \leq V(\mathbf{r}, t; c_h)$, proving the weak bound. If $T(\mathbf{r}, t) > 0$, then $c_h < c^*$ and condition (*) can be strengthened to a strict inequality, leading to a strict bound. Q.E.D.

Figure 5 ■ The refinancing cost bound on the market value of the loan.



The intuition underlying Proposition 7 is illustrated geometrically in Figure 5 for GNMAs (*i.e.*, $P(t) \equiv 0$). The refinancing cost, $V^s(\mathbf{r}, t; c) - V(\mathbf{r}, t; c)$ is increasing in the coupon payment so it is at least as large at c_h as at cm. However, the refinancing boundary defining c^* implies that $V^s(\mathbf{r}, t; c)$ increases by only $T(\mathbf{r}, t) + P(t)$ over the region from cm to c^*. But $V^s(\mathbf{r}, t; c)$ is increasing in the coupon payment from c_h to c^*, so the increase in $V^s(\mathbf{r}, t; c)$ from cm to c_h is less than $T(\mathbf{r}, t) + P(t)$. Because $V(\mathbf{r}, t; c)$ rises less rapidly than $V^s(\mathbf{r}, t; c)$, the increase in $V(\mathbf{r}, t; c)$ is less than $T(\mathbf{r}, t) + P(t)$. Exploiting $V(\mathbf{r}, t; cm) = F$ (by construction of cm), it follows that $V(\mathbf{r}, t; c) < F + T(\mathbf{r}, t) + P(t)$.

Solution Algorithm

In order to value the marketed loan one must simultaneously value the refinancing strategy and determine the contract rate on an equivalent new issue. The debt contracts can be valued by backward recursion from the common maturity date. At a given date, the lowest coupon debt contract is first valued at all points in the state space. Then the value of the next higher coupon contract is obtained at all points in the state space at the same date and this process is continued until all contracts have been valued. We then keep moving backward one date at a time and repeat the procedure, again valuing the debt contracts in the order of lowest to highest coupon (by proceeding in this order we will have examined previously any optimal refinancing opportunity for the borrower when a coupon is studied). In moving through the state space and time domain we need to value the strategy of the consumer (reflecting the cash flows faced by that consumer, including the refinancing costs on the current and subsequently optimal contracts) to determine the optimal refinancing program. The optimal refinancing strategy is then used to value the marketed cash flows of each particular security, which do not include the refinancing costs. If the transactions costs depended upon the loan balance, then the valuation would depend upon the loan balance at each point in time and, consequently, upon the sequence of previous refinancings. To avoid this path dependency, we assume in this section that the transaction costs are proportional to the contemporaneous loan balance. Under this assumption the optimal refinancing strategy is independent of the loan balance. Therefore, under proportional transactions costs the valuation is proportional to the outstanding balance, as it would be in the case of zero transactions costs.

One can solve simultaneously for the value of the refinancing strategy and the loan by backward recursion from the maturity date, using fine discrete grids for the time and state variables and a fine set of coupon payments, $c_o, \ldots, c_n$. For valuing any given coupon we need only the value of smaller coupons because of the possibility of refinancing only to a lower-rate instrument. Therefore, at a given t and r the problem is solved in order of increasing coupon payments. For illustration we assume that the interest rate dynamics are characterized by a single state variable for the description of the algorithm. At $t = 0$ we define $V(r, 0; c) = V^s(r, 0; c) = F(0; c)$ by the terminal condition. Then at $t = 1$ we begin by solving $V^s(r, 1; c_o)$ for all r, where c_o is low enough that it is never optimal to refinance the c_o coupon and, therefore, $V(r, 1; c_o) = V^s(r, 1; c_o)$ for all r. We then set $cm(r, 1) = c_o$ for any state r in which a refinanced contract would be offered at rate c_o. To the extent that the coupon grid is not fine enough and the refinancing rate, $cm(r, 1)$, lies off the coupon grid, we let $D(r, t = 1) = F(1; c_o) - V(r, 1; c_o)$ represent the resulting discount (or points) on the debt contract. The discount is included in the refinancing costs. We then solve for the refinancing strategy $V^s(r, 1; c_1)$ at all r. Beginning with $r = 0$ we check whether $V^s(r, 1; c_1) - V^s(r, 1; cm) > T(r, 1) + P(1)$ and if so we set $V^s(r, 1; c_1) = V^s(r, 1; cm) + T(r, 1) + P(1)$. Once we reach a state for which $V^s(r, 1; c_1) \leq V^s(r, 1; cm) + T(r, 1) + P(1)$ it cannot be optimal to refinance at higher interest rates. Through this process we determine the critical refinancing state, r_1, for the given coupon level c_1. We then solve for $V(r, t; c_1)$. In particular, in the refinancing region $r \leq r_1$ we have the boundary condition $V(r, t; c_1) = F(1; c_1) + P(1)$, and for $r > r_1$ we solve the partial differential equation for $V(r, t; c_1)$. Thus, by solving for $V^s(r, t; c_1)$ we determine the refinancing boundary for $V(r, t; c_1)$.

After having solved for $V(r, 1; c_1)$ we set $cm(r, 1) = c_1$ for any states where a c_1 contract would be offered and set the resulting discount, that is, $D(r, 1) = F(1; c_1) - V(r, 1; c_1)$ at those states. For example, $cm(r, t)$ and $D(r, t)$ could be set in this fashion for every state for which it did not pay to refinance the c_1 contract (this procedure is inefficient because the resetting of $cm(r, t)$ and $D(r, t)$ would occur for many states which would later be again reset for higher coupons because c_1 is not feasible for high values of r). We then repeat the entire procedure successively for higher coupons and then for each earlier date.

The valuations $V^s(r, t; c)$ and $V(r, t; c)$ contain all the information required to solve $V^s(r, t + 1; c)$ and $V(r, t + 1; c)$, so the problem can be solved by backward recursion.[13]

Discussion of Observed Prices

Observed prices on GNMA mortgage-backed securities can be compared to the predictions of the model. In particular, our theoretical results suggest that if borrowers prepay as quickly as our theory predicts in the premium region, then the value of the mortgage-backed securities will not exceed the refinancing costs (there are no prepayment penalties for GNMAs) plus the face value. Yet in the mid-1980s, 16% coupon GNMAs sold at a 14% premium over par when the new money coupon was approximately 10–12%. In fact, the actual refinancing costs were often 3–6% (including title and legal expenses as well as points paid in advance to the servicer).

The substantial violations of the pricing bound suggests that the market does not anticipate extremely rapid prepayment of the highest coupon rate instruments. Prepayments on an annualized basis have rarely exceeded 30–60%, even for the highest coupon instruments. This limits the degree to which the highest

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[13] Stanton and Wallace (1998) implement an extension of this algorithm.

coupons are pushed down toward face value due to expected refinancing in the near future of many of the loans. In fact, while the theory suggests that the value of higher-rate coupons can decline in the coupon, the data at most points in time indicates only a gradual flattening in the relationship between the market price and the coupon rate. This is illustrated by Figure 1. The higher coupon contracts are also less sensitive to interest rate changes because of the adjustment in prepayment rates.

The refinancing cost bound on the market values of loans can be extended to the case in which the borrowers are heterogeneous in their transaction costs or move likelihoods. The market value of a loan pool reflects the aggregation of such instruments, if the borrowers are diverse.[14] In such situations each type of borrower would have an idiosyncratic V^s curve. This curve must increase by exactly the borrower's T from the aggregate refinancing coupon (at this point the pool of loans to heterogeneous borrowers is worth par) to the borrower's specific optimal refinancing point. For each borrower the idiosyncratic value of his marketed loan rises less rapidly (*i.e.*, by less than that borrower's T) to its maximum (and then falls). The market value of a mortgage pool with diverse borrowers is an average of the values of the marketed loans for borrowers who have not already refinanced. Such an average rises by less than the maximum transaction costs across borrowers from the aggregate refinancing coupon. In this sense the transaction cost bound can be extended to heterogeneous borrowers.

In light of this bound, rationalizing observed GNMA prices require slow prepayment behavior by many of the borrowers who have not yet refinanced. We let $1 - \infty$ denote the fraction of borrowers who follow the theoretical refinancing strategy, V^p the value of a pool (per dollar face value) and V^{nc} the value of a loan in which refinancing is indefinitely deferred (the most extreme form of slow prepayment). Then, our bound implies

$$V^p \leq (1-\alpha)(T+1) + \alpha V^{nc},$$

or

$$\alpha \geq \frac{\mathrm{V^p - T - 1}}{\mathrm{V^{nc} - T - 1}}.$$

The critical value is increasing in V^p and decreasing in V^{nc} and T. The sample values in Table 1 illustrate that the market pricing has reflected the anticipation that many of the remaining borrowers will not follow the theoretical refinancing policy. Of course, less extreme behavior of some of the passive investors would

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[14] An alternative way to introduce heterogeneity is to permit refinancing costs to differ across borrowers (see Timmis 1985).

Table 1 ■ Minimum fraction of nonrefinancing borrowers required to explain violations of the refinancing cost bound.

Value of Scheduled Cash Flows if Never Refinancing	Implied Yield	Minimum Fraction of Nonrefinancing Borrowers
Refinancing costs are 3%		
130	12.56%	0.407
135	11.98%	0.344
140	11.43%	0.297
145	10.91%	0.262
150	10.43%	0.234
155	9.97%	0.212
160	9.54%	0.193
Refinancing costs are 6%		
130	12.56%	0.333
135	11.98%	0.276
140	11.43%	0.235
145	10.91%	0.205
150	10.43%	0.182
155	9.97%	0.163
160	9.54%	0.148

Example—16% coupon amortizing loan with 25 years remaining to maturity, selling at a 14% premium.

imply an even higher percentage of borrowers following approximately the theoretical refinancing policy. To the extent that much of the mortgage pool had already prepaid, such slow prepayment behavior is only required for a smaller fraction of the original population of borrowers. Of course, an important reason borrowers could prepay slowly is the inability of much of the remaining pool ("survivors") to qualify for refinancing.

Concluding Comments

We have developed an approach for valuing debt contracts in the presence of costly refinancing and have analyzed the qualitative nature of debt valuation with costly refinancing. We derive a strong bound on the pricing of debt contracts with refinancing costs. This bound and our framework for valuation with refinancing costs applies even if the borrower continues to refinance into contracts with transaction costs. Passive behavior by some borrowers or the inability of some borrowers to requalify for new financing may help explain the pricing.[15]

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[15] An interesting reduced-form analysis of empirical prepayment behavior is in Richard and Roll (1989). Bennett, Peach and Peristiani (2001) suggest that effective transaction costs have been declining over time and that, in addition, the borrower's refinancing behavior has become more sensitive to interest rate movements.

An interesting extension of the analysis would incorporate the effect of heterogeneity in borrower information. For example, even if borrowers have different probabilities of prepaying due to moving in the future (*e.g.*, if it is optimal to transfer the property and move before maturity), the coupon available to all borrowers is identical and depends upon the distribution of borrower characteristics. Then the problems of different types of borrowers must be solved simultaneously because the refinancing rate available to borrowers depends upon the refinancing strategy of the population in which they are included. The analysis is more complex in the presence of heterogeneous information because the valuation problem requires simultaneous solution of the optimal repayment decision for each type of consumer and the refinancing rate on new contracts that would prevail in the aggregate in such an environment. Mortgage borrowers can differ in their probability of a prepayment because of differences in their utility gain from prepaying and transferring the property (see Dunn and Spatt 1985, 1988). Aggregating the effect of the diverse refinancing decisions of different borrowers is required for the market valuation of each loan pool and determination of the contract rate on newly issued debt. The new contract rate is simultaneously determined with the refinancing decisions. Of course, prepayments by those with the highest probability of prepaying influences the subsequent conditional distribution of prepayment probabilities among those who have not prepaid. Even when it is optimal for many investors to refinance, those who anticipate a forced liquidation in a short period of time can be reluctant to pay refinancing costs that might need to be reincurred shortly. To a degree this can justify some of the observed premiums on the high coupon mortgage-backed securities.

The research identifies a mechanism to reduce the costs of debt contracting. The conventional fixed-rate callable instrument with a costly refinancing option can be dominated by a security in which the borrower has the option to reduce the coupon rate to the market level by paying a predetermined fee directly to the lender. By including the refinancing option as part of the agreement, "refinancing" can occur without entering a new contract or paying fees to third parties. If the conventional refinancing costs are specified as part of the call premium, then the contract rate is lower than if the costs are paid out to third parties. In fact, this observation suggests a puzzling aspect of the observed contractual arrangements.

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